Generalized functions in mathematical physics.

Main ideas and concepts

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Contents

Preface

Presently, the notion of function is not as finally crystallized and definitely established as it seemed at the end of the 19th century; one can say that at present this notion is still in evolution, and that the dispute concerning the vibrating string is still going on only, of course, in different scientific circumstances, involving other personalities and using other terms.

Luzin N.N. (1935) [**42**]

It is symbolic that in that same year of 1935, S.L. Sobolev, who was 26 years old that time, submitted to the editorial board of the journal "Matematicheskiy sbornik" his famous work [**61**] and published at the same time its brief version in "Doklady AN SSSR" [**60**]. This work laid foundations of a completely new outlook on the concept of function, unexpected even for N.N. Luzin — the concept of a generalized function (in the framework of the notion of distribution introduced later). It is also symbolic that the work by Sobolev was devoted to the Cauchy problem for hyperbolic equations and, in particular, to the same vibrating string.

In recent years Luzin's assertion that the discussion concerning the notion of function is continuing was confirmed once again, and the stimulus for the development of this fundamental concept of mathematics is, as it was before, the equations of mathematical physics (see, in particular, Addition written by Yu.V. Egorov and [**10, 11, 16, 17, 18, 32, 49, 67**]). This special role of the equations of mathematical physics (in other words, partial differential equations directly connected with natural phenomena) is explained by the fact that they express the mathematical essence of the fundamental laws of the natural sciences and consequently are a source and stimulus for the development of fundamental mathematical concepts and theories.

The crucial role in appearance of the theory of generalized functions (in the sense the theory of distributions) was played by J. Hadamard, K.O. Friedrichs, S. Bochner, and especially to L. Schwartz, who published, in 1944–1948, a series of remarkable papers concerning the theory of distributions, and in 1950–1951 a two-volume book [**54**], which immediately became classical. Being a masterpiece and oriented to a wide circle of specialists, this book attracted the attention of many people to the theory of distributions. The huge contribution to its development was made by such prominent mathematicians as I.M. Gel'fand, L. Hörmander and many others. As a result, the theory of distributions has changed all modern analysis and first of all the theory of partial differential equations. Therefore, the foundations of the theory of distributions became necessary for general education of physicists and mathematicians. As for students specializing in the equations of mathematical physics, they cannot even begin any serious work without knowing the foundations of the theory of distributions.

Thus, it is not surprising that a number of excellent monographs and textbooks (see, for example, [**12, 22, 23, 25, 31, 40, 44, 54, 57, 59, 62, 68, 69**]) are devoted to the equations of mathematical physics and distributions. However, most of them are intended for rather well prepared readers. As for this small book, I hope that it will be clear even to undergraduates majoring in physics and mathematics and will serve to them as starting point for a deeper study of the above-mentioned books and papers.

In a nutshell the book gives an interconnected presentation of a some basic ideas, concepts, results of the theory of generalized functions (first of all, in the framework of the theory of distributions) and equations of mathematical physics.

Chapter 1 acquaints the reader with some initial elements of the language of distributions in the context of the classical equations of mathematical physics (the Laplace equation, the heat equation, the string equation). Here some basic facts from the theory of the Lebesgue integral are presented, the Riesz spaces of integrable functions are introduced. In the section devoted to the heat equation, the student of mathematics can get familiar with the method of dimensionality and similarity, which is not usually included in the university program for mathematicians, but which is rather useful on the initial stage of study of the problems of mathematical physics.

Chapter 2 is devoted to the fundamentals of the theory of distributions due to L. Schwartz. Section 16 is the most important. The approach to some topics can also be interesting for the experts.

Chapter 3 acquaints the reader with some modern tools and methods for the study of linear equations of mathematical physics. The basics of the theory of Sobolev spaces, the theory of pseudodifferential operators, the theory of elliptic problems (including some elementary results concerning the index of elliptic operators) as well as some other problems connected in some way with the Fourier transform (ordinary functions and distributions) are given here.

Now I would like to say a few words concerning the style of the book. A part of the material is given according to the scheme: definition — theorem — proof. This scheme is convenient for presenting results in clear and concentrated form. However, it seems reasonable to give a student the possibility not only to study *a priori* given definitions and proofs of theorems, but also to discover them while considering the problems involved. A series of sections serves this purpose. Moreover, a part of the material is given as exercises and problems. Thus, reading the book requires, in places, a certain effort. However, the more difficult problems are supplied with hints or references. Problems are marked by the letter **P** (hint on Parking for the solution of small Problems).

The importance of numerous notes is essentially connected with a playful remark by V.F. D'yachenko: "The most important facts should be written in notes, since only those are read". The notes are typeset in a small font and located in the text immediately after the current paragraph.

I am very grateful to Yu.V. Egorov, who kindly agreed to write Addendum to the book. I would also like to acknowledge my gratitude to M.S. Agranovich, A.I. Komech, S.V. Konyagin, V.P. Palamodov, M.A. Shubin, V.M. Tikhomirov, and M.I. Vishik for the useful discussions and critical remarks that helped improve the manuscript. I am also thankful to E.V. Pankratiev who translated this book and produced the CRC.

While preparing this edition, some corrections were made and the detected misprints have been corrected.

A.S. Demidov

Notation

$\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{Z} = \{0, \pm1, \pm2, \dots\}$ and $\mathbb{Z}_+ = \{0, \pm1, \pm2, \dots\}$ are the sets of natural numbers, integers and non-negative integers.

$X \times Y$ is the Cartesian product of the sets X and Y, $X^n = X^{n-1} \times X$.

$i = \sqrt{-1}$ is the imaginary unit ("dotted i").

$\mathring{i} = 2\pi i$ the imaginary 2π ("i with a circle").

$\mathbb{R}^n$ and $\mathbb{C}^n$ are n-dimensional Euclidean and complex spaces; $\mathbb{R} = \mathbb{R}^1$, $\mathbb{C} = \mathbb{C}^1 \ni z = x + iy$, where $x = \Re z \in \mathbb{R}$ and $y = \Im z \in \mathbb{R}$.

$x < y$, $x \leq y$, $x > y$, $x \geq y$ are the order relations on $\mathbb{R}$.

$a \gg 1$ means the a sufficiently large.

$\{x \in X \mid P\}$ is the set of elements which belong to X and have a property P.

$]a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$; $[a, b]$, $]a, b[$ and $[a, b[$ are defined similarly.

$\{a_n\}$ is the sequence $\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \dots\}$.

$f : X \ni x \mapsto f(x) \in Y$ is the mapping $f : X \to Y$, putting into correspondence to an element $x \in X$ the element $f(x) \in Y$.

1_A is the characteristic function of the set A, i.e. $1_A = 1$ in A and $1_A = 0$ outside of A.

$\operatorname{arccot} \alpha = \frac{\pi}{2} - \arctan \alpha$.

$x \to a$ means that the numerical variable x converges (tends) to a.

$\Longrightarrow$ means "it is necessary follows".

$\Longleftrightarrow$ means "if and only if" ("iff"), i.e an equivalence.

$A \Subset \Omega$ means A is compactly embedded in Ω (see Definition 3.2).

$C^m(\Omega)$, $C_b^m(\Omega)$, $C^m(\bar{\Omega})$, $PC^m(\Omega)$, $PC_b^m(\Omega)$, $C_0^m(\Omega)$, $C_0^m(\bar{\Omega})$ see Definition 3.1 (for $0 \leq m \leq \infty$).

$L^p(\Omega)$, $L^\infty(\Omega)$, $L^p_{loc}(\Omega)$ see Definitions 9.1, 9.9, 9.15.

$\mathcal{D}^\flat(\Omega)$, $\mathcal{D}^\#(\Omega)$, $\mathcal{D}(\Omega)$, $\mathcal{D}'(\Omega)$ see Definitions 12.2, 13.1, 16.7, 16.9.

$\mathcal{E}(\Omega)$, $\mathcal{E}'(\Omega)$ see **P**.16.13.

$\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$ see Definitions 17.10, 17.18.

CHAPTER 1

Introduction to problems of mathematical physics

1. Temperature at a point? No! In volumes contracting to the point

Temperature. We know this word from our childhood. The temperature can be measured by a thermometer... This first impression concerning the temperature is, in a sense, nearer to the essence than the representation of the temperature as a function of a point in space and time. Why? Because the concept of the temperature as a function of a point arose as an abstraction in connection with the conception of continuous medium. Actually, a physical parameter of the medium under consideration (for instance, its temperature) is first measured by a device in a "large" domain containing the fixed point ξ, then using a device with better resolution in a smaller domain (containing the same point) an so on. As a result, we obtain a (finite) sequence of numbers $\{a_1, \dots, a_M\}$ — the values of the physical parameter in the sequence of embedded domains containing the point ξ. We idealize the medium considered, by assuming that the construction of the numerical sequence given above is for an infinite system of domains containing the point ξ and embedded in each other. Then we obtain an infinite numerical sequence $\{a_m\}$. If we admit (this is the essence of the conception of the continuous medium[1]), that such a sequence exists and has a limit (which does not depend on the choice of the system of embedded in each other domains), then this limit is considered as the value of the physical parameter (for instance, temperature) of the considered medium at the point ξ.

[1] In some problems of mathematical physics, first of all in nonlinear ones, it is reasonable (see, for example, [**10, 11, 16, 18, 32, 49**]) to

consider a more general conception of the continuous medium in which a physical parameter (say, temperature, density, velocity…) is characterized not by the values measured by one or another set of "devices", in other words, not by a functional of these "devices", but a "convergent" sequence of such functionals which define, similarly to nonstandard analysis [**5, 13, 73**], a thin structure of a neighbourhood of one or another point of continuous medium.

Thus, the concept of continuous medium occupying a domain[2)] Ω, assumes that the numerical characteristic f of a physical parameter considered in this domain (i.e. in Ω) is a function in the usual sense: a mapping from the domain Ω into the numerical line (i.e. into $\mathbb{R}$ or into $\mathbb{C}$). Moreover, the function f has the following property:

$$\langle f, \varphi_m^\xi \rangle = a_m, \quad m = 1, \dots, M. \tag{1.1}$$

Here, a_m are the numbers introduced above, and the left-hand side of (1.1), which is defined[3)] by the formula $\langle f, \varphi^\xi \rangle = \int f(x)\varphi^\xi(x)dx$, represents the "average" value of the function f, measured in the neighbourhood of the point $\xi \in \Omega$ by using a "device", which will be denoted by $\langle \cdot, \varphi^\xi \rangle$. The "device" has the resolving power, that is determined by its "device function" (or we may also say "test function") $\varphi^\xi : \Omega \to \mathbb{R}$. This function is normed: $\int \varphi^\xi(x)dx = 1$.

[2)] Always below, if the contrary is not said explicitly, the domain Ω is an open connected set in $\mathbb{R}^n$, where $n > 1$, with a sufficiently smooth $(n-1)$-dimensional boundary $\partial\Omega$.

[3)] Integration of a function g over a fixed (in this context) domain will be often written without indication of the domain of integration, and sometimes simply in the form $\int g$.

Let us note that more physical are "devices", in which φ^ξ has the form of a "cap" in the neighbourhood of the point ξ, i.e. $\varphi^\xi(x) = \varphi(x-\xi)$ for $x \in \Omega$, where the function $\varphi : \mathbb{R}^n \ni x = (x_1, \dots, x_n) \longmapsto \varphi(x) \in \mathbb{R}$ has the following properties:

$$\begin{gathered} \varphi \geq 0, \ \int \varphi = 1, \\ \varphi = 0 \text{ outside the ball } \{x \in \mathbb{R}^n \big| \ |x| \equiv \sqrt{x_1^2 + \dots + x_n^2} \leq \rho\}. \end{gathered} \tag{1.2}$$

Here, $\rho \leq 1$ is such that $\{x \in \Omega \ \big| \ |x-\xi| < \rho\} \subset \Omega$. Often one can assume that the "device" measures the quantity f uniformly in the domain $\omega \in \Omega$. In this case, $\varphi = 1_\omega/|\omega|$, where 1_ω is the characteristic function of the domain ω (i.e. $1_\omega = 1$ in ω and $1_\omega = 0$ outside ω), and $|\omega|$ is the volume of the domain ω (i.e. $|\omega| = \int 1_\omega$). In particular, if $\Omega = \mathbb{R}^n$ and $\omega = \{x \in \mathbb{R}^n \big| \, |x| < \alpha\}$, then $\varphi(x) = \delta_\alpha(x)$, where

$$\delta_\alpha(x) = \begin{cases} \alpha^{-n}/|B_n| & for\ |x| \le \alpha, \\ 0 & for\ |x| > \alpha, \end{cases} \tag{1.3}$$

and $|B_n|$ is the volume of the unit ball B_n in $\mathbb{R}^n$.

1.1.**P.** *It is well known that* $|B_2| = \pi$ *and* $|B_3| = 4\pi/3$. *Try to calculate* $|B_n|$ *for* $n > 3$. *We shall need it below.*

Hint. Obviously, $|B_n| = \sigma_n/n$, where σ_n is the area of the surface of the unit $(n-1)$-dimensional sphere in $\mathbb{R}^n$, since $|B_n| = \int_0^1 r^{n-1}\sigma_n dr$. If the calculation of σ_n for $n > 3$ seems to the reader difficult or noninteresting, he can read the following short and unexpectedly beautiful solution.

Solution. We have

$$\left(\int_{-\infty}^{\infty} e^{-t^2} dt\right)^n = \int_{\mathbb{R}^n} e^{-|x|^2} dx = \int_0^{\infty} e^{-r^2} r^{n-1}\sigma_n dr = (\sigma_n/2)\cdot\Gamma(n/2), \tag{1.4}$$

where $\Gamma(\cdot)$ is the Euler function defined by the formula

$$\Gamma(\lambda) = \int_0^{\infty} t^{\lambda-1}e^{-t}dt, \text{ where } \Re\lambda > 0. \tag{1.5}$$

For $n = 2$ the right-hand side of (1.4) is equal to π. Therefore,

$$\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}. \tag{1.6}$$

Thus, $\sigma_n = 2\pi^{n/2}\Gamma^{-1}(n/2)$. By taking $n = 3$, we obtain $2\Gamma(3/2) = \sqrt{\pi}$. By virtue of the remarkable formula $\Gamma(\lambda + 1) = \lambda \cdot \Gamma(\lambda)$, (obtained from (1.5) by integration by parts and implying $\Gamma(n+1) = n!$), this implies that $\Gamma(1/2) = \sqrt{\pi}$. Now we get that

$$\sigma_{2n} = \frac{2\pi^n}{(n-1)!}, \qquad \sigma_{2n+1} = \frac{2\pi^n}{(n-1/2)\cdot(n-3/2)\cdot\dots\cdot 3/2\cdot 1/2}. \tag{1.7}$$

□

2. The notion of δ-sequence and δ-function

In the preceding section the idea was indicated that the definition of a function $f : \Omega \to \mathbb{R}$ (or of a function $f : \Omega \to \mathbb{C}$) as a mapping from a domain $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}$ (or into $\mathbb{C}$) is equivalent to determination of its "average values":

$$\langle f, \varphi\rangle = \int\limits_{\Omega} f(x)\varphi(x)\,dx, \qquad \varphi \in \Phi, \tag{2.1}$$

where Φ is a sufficiently "rich" set of functions on Ω. A sufficiently general result concerning this fact is given in Section 10. Here, we prove a simple but useful lemma. Preliminary, we introduce for $\varepsilon \in]0,1]$ the function $\delta_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ by the formula

$$\delta_\varepsilon(x) = \varphi(x/\varepsilon)/\varepsilon^n,\ \varphi \geq 0,\ \int \varphi = 1,\ \varphi = 0 \text{ outside } B_n. \tag{2.2}$$

Let us note that for $1/\varepsilon \gg 1$ we have

$$\int\limits_{\mathbb{R}^n} \delta_\varepsilon(x)\,dx = \int\limits_{\Omega} \delta_\varepsilon(x-\xi)\,dx = 1, \quad \xi \in \Omega.$$

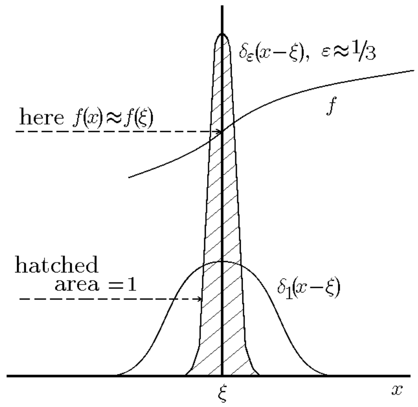
2.1. Lemma. *Let $f \in C(\Omega)$, i.e. f is continuous in $\Omega \subset \mathbb{R}^n$. Then*

$$f(\xi) = \lim_{\varepsilon\to 0} \int\limits_{\Omega} f(x)\delta_\varepsilon(x-\xi)\,dx, \quad \xi \in \Omega, \tag{2.3}$$

i.e. the function f can be recovered by the family of "average values"

$$\left\{\int f(x)\cdot\delta_\varepsilon(x-\xi)dx\right\}_{\xi\in\Omega,\ \varepsilon>0}.$$

Proof. For any $\eta > 0$, there exists $\varepsilon > 0$ such that $|f(x) - f(\xi)| \leq \eta$ if $|x-\xi| \leq \varepsilon$. Therefore,



$$\left|\left[\int_\Omega f(x)\delta_\epsilon(x-\xi)\,dx\right]-f(\xi)\right| = \left|\int_\Omega (f(x)-f(\xi))\delta_\epsilon(x-\xi)\,dx\right|$$
$$\leq \int_{|x-\xi|\leq\epsilon} |f(x)-f(\xi)|\delta_\epsilon(x-\xi)\,dx$$
$$\leq \eta \int_{|x-\xi|\leq\epsilon} \delta_\epsilon(x-\xi)\,dx = \eta.$$

□

2.2. Definition. Let Φ be a subspace of the space $C(\Omega)$ and $\xi\in\Omega$. A sequence $\{\delta_\epsilon(x-\xi), x\in\Omega\}_{\epsilon\in\mathbb{R},\epsilon\to 0}$ of functions $x\longmapsto \delta_\epsilon(x-\xi)$ such that equality (2.3) holds for any $f\in C(\Omega)$ (for any $f\in\Phi$) is called δ-*sequence* (on the space[1] Φ) concentrated near the point ξ. The last words are usually skipped.

[1] The notion of δ-sequence on the space Φ allow to obtain a series of rather important results. Some of them are mentioned at the beginning of Section 4.

In Section 4 some examples of δ-sequences on one or another subspace $\Phi\subset C(\Omega)$ are given. Important examples of such sequences are given in Section 3.

2.3. Definition. A linear functional[2] δ_ξ defined on the space $C(\Omega)$ by the formula

$$\delta_\xi : C(\Omega) \ni f \longmapsto f(\xi) \in \mathbb{R} \text{ (or } \mathbb{C}), \ \xi \in \Omega, \tag{2.4}$$

is called the *δ-function*, or the *Dirac function* concentrated at the point ξ.

[2] A (linear) functional on a (linear) space of functions is defined as a (linear) mapping from this functional space into a number set.

Often one writes δ-function (2.4) in the form $\delta(x-\xi)$ and its action on a function $f \in C(\Omega)$ writes (see formula (1.1)) in the form

$$\langle f(x), \delta(x-\xi)\rangle = f(\xi) \quad \textit{or} \quad \langle \delta(x-\xi), f(x)\rangle = f(\xi). \tag{2.5}$$

The following notation is also used: $\langle f, \delta_\xi\rangle = f(\xi)$ or $\langle \delta_\xi, f\rangle = f(\xi)$. The Dirac function can be interpreted as a measuring instrument at a point (a "thermometer" measuring the "temperature" at a point). If $\xi = 0$, then we write δ or $\delta(x)$.

3. Some spaces of smooth functions. Partition of unity

The spaces of smooth functions being introduced in this section play very important role in the analysis. In particular, they give examples of the space Φ in the "averaging" formula (2.1).

3.1. Definition. Let Ω be an open set in $\mathbb{R}^n$, $\bar{\Omega}$ the closure of Ω in $\mathbb{R}^n$, and $m \in \mathbb{Z}_+$, i.e., m is a non-negative integer. Then[1]

[1] If $m = 0$, then the index m in the designation of the spaces defined below is usually omitted.

3.1.1. $C^m(\Omega)$ (respectively, $C_b^m(\Omega)$) is the *space! $C^m(\Omega)$ of m-times continuously differentiable* (respectively, *with bounded derivatives*) *functions* $\varphi : \Omega \to \mathbb{C}$, i.e., such that the function $\partial^\alpha\varphi$ is continuous (and respectively, bounded) in Ω for $|\alpha| \le m$. Here and below

$$\partial^\alpha\varphi(x) = \frac{\partial^{|\alpha|}\varphi(x)}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}, \ |\alpha| = \alpha_1+\dots+\alpha_n, \ \alpha_j \in \mathbb{Z}_+ = \{0,1,2,\dots\}.$$

The vector $\alpha = (\alpha_1, \dots, \alpha_n)$ is called a multiindex.

3.1.2. $C^m(\bar{\Omega}) = C^m(\mathbb{R}^n)\big|_{\Omega}$, i.e.[2] $C^m(\bar{\Omega})$ is the *restriction of the space* $C^m(\mathbb{R}^n)$ to $\bar{\Omega}$. This means that $\varphi \in C^m(\bar{\Omega}) \iff$ there exists a function $\psi \in C^m(\mathbb{R}^n)$ such that $\varphi(x) = \psi(x)$ for $x \in \Omega$.

[2] The space $C^m(\bar{\Omega})$, in general, does not coincide with the *space of functions m-times continuously differentiable up to the boundary*. However, they coincide, if the boundary of the domain is sufficiently smooth.

3.1.3. $PC^m(\Omega)$ (respectively, $PC^m_b(\Omega)$) is the *space of functions m-times piecewise continuously differentiable* (*and* respectively, *bounded*) *in* Ω; this means that $\varphi \in PC^m(\Omega)$ (respectively, $\varphi \in PC^m_b(\Omega)$) if and only if the following two conditions are satisfied. First, $\varphi \in C^m(\Omega \setminus K_0)$ (respectively, $\varphi \in C^m_b(\Omega \setminus K_0)$) for a compact[3] $K_0 \subset \Omega$. Second, for any compact $K \subset \bar{\Omega}$ there exists a finite number of domains $\Omega_j \subset \Omega$, $j = 1, \dots, N$, each of them is an intersection of a finite number of domains with smooth boundaries, such that $K \subset \bigcup_{j=1}^N \bar{\Omega}_j$ and $\varphi\big|_{\omega} \in C^m(\bar{\omega})$ for any connected component ω of the set $\left(\Big(\bigcup\limits_{j=1}^N \Omega_j\Big) \setminus \Big(\bigcup\limits_{j=1}^N \partial\Omega_j\Big)\right)$.

[3] A set $K \subset \mathbb{R}^n$ is called *compact*, if K is bounded and closed.

3.1.4. The *support* of a function $\varphi \in C(\Omega)$, denoted by $\operatorname{supp}\varphi$, is the complement in Ω of the set $\{x \in \Omega \mid \varphi(x) \neq 0\}$. In other words, $\operatorname{supp}\varphi$ is the smallest set closed in Ω such that the function φ vanishes outside this set.

3.1.5. $C^m_0(\bar{\Omega}) = \{\varphi \in C^m(\bar{\Omega}) \mid \operatorname{supp}\varphi \text{ is a compact}\}$.

3.1.6. $C^m_0(\Omega) = \{\varphi \in C^m_0(\bar{\Omega}) \mid \operatorname{supp}\varphi \subset \Omega\}$.

3.1.7. $C^\infty(\Omega) = \bigcap\limits_m C^m(\Omega), \dots, C^\infty_0(\Omega) = \bigcap\limits_m C^m_0(\Omega)$.

3.1.8. If $\varphi \in C^m_0(\Omega)$ (or $\varphi \in C^\infty_0(\Omega)$) and $\operatorname{supp}\varphi \subset \omega$, where ω is a subdomain of Ω, then the function φ is identified with its restriction to ω. In this case we write: $\varphi \in C^m_0(\omega)$ (or $\varphi \in C^\infty_0(\omega)$).

3.2. Definition. We say that a set A is *compactly embedded* in Ω, if $\bar{A}$ is a compact and $\bar{A} \subset \Omega$. In this case we write $A \Subset \Omega$.

Obviously, $C_0^m(\Omega) = \{\varphi \in C^m(\Omega) \mid \operatorname{supp}\varphi \Subset \Omega\}$, and

$$C_0^m(\Omega) \subsetneq C_0^m(\bar{\Omega}) \subsetneq C^m(\bar{\Omega}) \subsetneq C^m(\Omega) \subsetneq PC^m(\Omega),$$

where the first inclusion and the third one should be replaced by $=$, if $\Omega = \mathbb{R}^n$.

3.3. Example.

$$C_0^m(\mathbb{R}^n) \ni \varphi : x \longmapsto \varphi(x) = \begin{cases} (1-|x|^2)^{m+1} & for\ |x| < 1, \\ 0 & for\ |x| \geq 1. \end{cases}$$

3.4. Example.

$$C_0^\infty(\mathbb{R}^n) \ni \varphi : x \longmapsto \varphi(x) = \begin{cases} \exp(1/(|x|^2-1)) & for\ \ |x| < 1, \\ 0 & for\ \ |x| \geq 1. \end{cases}$$

3.5. Example (A special case of (2.2)). Let $\epsilon > 0$. we set

$$\varphi_\epsilon(x) = \varphi(x/\epsilon), \tag{3.1}$$

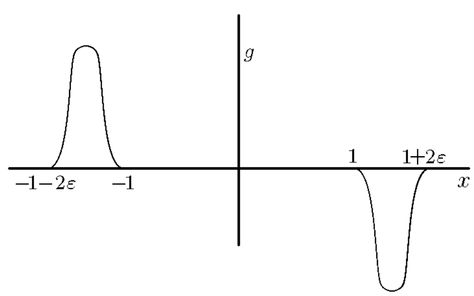
where φ is the function from Example 3.4. Then the function

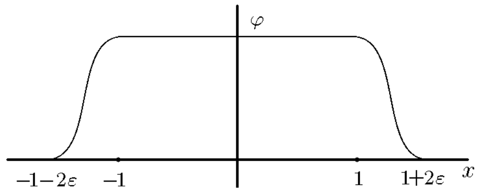
$$\delta_\epsilon : x \longmapsto \varphi_\epsilon(x) / \int\limits_{\mathbb{R}^n} \varphi_\epsilon(x) dx, \quad x \in \mathbb{R}^n, \tag{3.2}$$

belongs to $C_0^\infty(\mathbb{R}^n)$ and, moreover,

$$\delta_\epsilon(x) \geq 0,\ \forall x \in \mathbb{R}^n,\ \delta_\epsilon(x) = 0\ for\ |x| > \epsilon,\ \int\limits_{\mathbb{R}^n} \delta_\epsilon(x)\, dx = 1. \tag{3.3}$$

3.6. Example. Let $x \in \mathbb{R}$, $\varphi : x \longmapsto \varphi(x) = \int_{-\infty}^x g(\tau) d\tau$, where (see Fig.) $g(-x) = -g(x)$ and $g(x) = \delta_\epsilon(x+1+\epsilon)$ for $x < 0$ (δ_ϵ satisfies (3.3)).





We have $\varphi \in C_0^\infty(\mathbb{R})$, $0 \le \varphi \le 1$, $\varphi(x) = 1$ for $|x| < 1$.

3.7. EXAMPLE. Let $(x_1, \dots, x_n) \in \mathbb{R}^n$ be Euclidian coordinates of a point $x \in \mathbb{R}^n$. Taking φ from Example 3.6, we set

$$\psi_\nu(x) = \sum_{|k|=\nu} \varphi(x_1+2k_1)\cdot\dots\cdot\varphi(x_n+2k_n),\ k_j \in \mathbb{Z},\ |k| = k_1+\dots+k_n.$$

Then the family $\{\varphi_\nu\}_{\nu=0}^\infty$ of the functions $\varphi_\nu(x) = \psi_\nu(x)/\left(\sum_{\nu=0}^\infty \psi_\nu(x)\right)$ form a *partition of unity* in $\Omega = \mathbb{R}^n$, i.e., $\varphi_\nu \in C_0^\infty(\Omega)$, and

(1) for any compact $K \subset \Omega$, only a finite number of functions φ_ν is non-zero in K;
(2) $0 \le \varphi_\nu(x) \le 1$ and $\sum_\nu \varphi_\nu(x) = 1\ \forall x \in \Omega$.

3.8. PROPOSITION. *For any domain $\omega \Subset \Omega$, there exists a function $\varphi \in C_0^\infty(\Omega)$ such that $0 \le \varphi \le 1$ and $\varphi(x) = 1$ for $x \in \omega$.*

PROOF. Let $\epsilon > 0$ is such that 3ϵ is less than the distance from ω to $\partial\Omega = \bar{\Omega}\backslash\Omega$. We denote the ϵ-neighbourhood of ω by ω_ϵ. Then the function

$$x \longmapsto \varphi(x) = \int\limits_{\omega_\epsilon} \delta_\epsilon(x-y)dy, \quad x \in \Omega$$

(δ_ϵ from (3.2)) has the required properties. □

3.9. DEFINITION. Let $\{\Omega_\nu\}$ be a family of subdomains $\Omega_\nu \Subset \Omega$ of a domain $\Omega = \cup\Omega_\nu$. Suppose that any compact $K \Subset \Omega$ has a nonempty intersection with only a finite number of domains Ω_ν. Then we say that the family $\{\Omega_\nu\}$ forms a *locally finite cover* of Ω.

3.10. THEOREM (on partition of unity). *Let $\{\Omega_\nu\}$ be a locally finite cover of a domain Ω. Then there exists a partition of unity subordinate to a locally finite cover, i.e., there exists a family of functions $\varphi_\nu \in C_0^\infty(\Omega_\nu)$ that satisfies conditions (1)–(2) above.*

The reader can himself readily obtain the proof; see, for example, [**69**]. The partition of unity is a very common and convenient tool, by using which some problems for the whole domain Ω can be reduced to problems for subdomains covering Ω (see, in particular, in this connection Sections 11, 20, and 22).

4. Examples of δ-sequences

The examples in this section are given in the form of exercises. Exercise **P**.4.1 will be used below in the deduction of the Poisson formula for the solution of the Laplace equation (see Section 5), **P**.4.2 will be used for the Poisson formula for the solution of the heat equation (see Section 6), **P**.4.3 will be used in the proof of the theorem on the inversion of the Fourier transform (see Section 17), and with the help of **P**.4.3 the Weierstrass theorem on approximation of continuous functions by polynomials can be easily proved (see Section 19).

4.1.**P.** *Show that the sequence* $\{\delta_y\}_{y\to+0}$ *of the functions* $\delta_y(x) = \frac{1}{\pi}\frac{y}{x^2+y^2}$, *where* $x \in \mathbb{R}$, *is a* δ*-sequence on the space* $C_b(\mathbb{R})$ *(see Definition 3.1.1) and is not a* δ*-sequence on* $C(\mathbb{R})$.

4.2.**P.** *Show that the sequence* $\{\delta_t\}_{t\to+0}$ *of the functions* $\delta_t(x) = \frac{1}{2\sqrt{\pi t}}e^{-x^2/4t}$, *where* $x \in \mathbb{R}$, *is not a* δ*-sequence on* $C(\mathbb{R})$, *but is a* δ*-sequence on the space* $\Phi \subset C(\mathbb{R})$ *of the functions that satisfy the condition* $\forall\varphi \in \Phi\ \exists a > 0$ *such that* $|\varphi(x)\exp(-ax^2)| \to 0$ *for* $|x| \to \infty$.

4.3.**P.** *Show that the sequence* $\{\delta_\nu\}_{1/\nu\to 0}$ *of the functions* $\delta_\nu(x) = \frac{\sin \nu x}{\pi x}$, *where* $x \in \mathbb{R}$, *is a* δ*-sequence on the space* $\Phi \subset C^1(\mathbb{R})$ *of the functions* φ *such that*

$$\int\limits_{\mathbb{R}} |\varphi(x)|\, dx < \infty, \qquad \int\limits_{\mathbb{R}} |\varphi'(x)|\, dx < \infty,$$

4.4.**P.** *By taking the polynomials* $\delta_k(x) = \frac{k}{\sqrt{\pi}}\left(1 - \frac{x^2}{k}\right)^{k^3}$, *where* $x \in \mathbb{R}$ *and* k *is a positive integer, show that the sequence* $\{\delta_k\}_{1/k\to 0}$ *is a* δ*-sequence on the space* $C_0(\mathbb{R})$, *but is not a* δ*-sequence on the space* $C_b(\mathbb{R})$ *(compare* **P***.4.1)*

4.5. Remark. For exercises **P**.4.1–**P**.4.4 it is helpful to draw sketches of graphs of the appropriate functions. Exercises **P**.4.1–**P**.4.2 are simple enough, exercises **P**.4.3–**P**.4.4 are more difficult, because the corresponding functions are alternating. In Section 13 Lemma 13.10 is proven that allows us to solve readily **P**.4.3–**P**.4.4. While solving **P**.4.2–**P**.4.4, one should use the well-known equalities:

$$\int_{-\infty}^{\infty} e^{-y^2}\,dy = \sqrt{\pi}, \qquad \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi, \qquad \lim_{\nu\to\infty} (1 - a/\nu)^{\nu} = e^{-a}.$$

5. On the Laplace equation

Three pearls of mathematical physics. Rephrasing the title of the well-known book by A.Ya. Khinchin [**33**], one can say so about three classical equations in partial derivatives: the Laplace equation, the heat equation and the string equation. One of this pearls has been found by Laplace, when he analyzed[1)] Newton's gravitation law.

[1)] See in this connection Section 1 of the book by S.K. Godunov [**25**].

5.1. Definition. A function $u \in C^2(\Omega)$ is called *harmonic* on an open set $\Omega \subset \mathbb{R}^n$, if it satisfies in Ω the (homogeneous[2)]) *Laplace equation*

$$\Delta u = 0,\ \textit{where}\ \Delta : C^2(\Omega) \ni u \longmapsto \Delta u \equiv \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} \in C(\Omega), \tag{5.1}$$

and $x_1, \ldots, x_n$ are the Euclidian coordinates of the point $x \in \Omega \subset \mathbb{R}^n$. Operator[3)] (5.1) denoted by the Greek letter Δ — "delta" — is called the *Laplace operator* or *Laplacian*.

[2)] The equation $\Delta u = f$ with non-zero right-hand side is sometimes called the *Poisson equation*.

[3)] By an *operator* we mean a mapping $f : X \to Y$, where X and Y are functional spaces.

5.2. **P.** *Let a function* $u \in C^2(\Omega)$, *where* $x \in \Omega \subset \mathbb{R}^n$, *depend only on* $\rho = |x|$, *i.e.*, $u(x) = v(\rho)$. *Show that* Δu *also depends only on* ρ *and* $\Delta u = \frac{\partial^2 v(\rho)}{\partial \rho^2} + \frac{n-1}{\rho}\frac{\partial v(\rho)}{\partial \rho}$.

Harmonic functions of two independent variables are closely connected with *analytic functions of one complex variable*, i.e., with the functions

$$w(z) = u(x,y) + iv(x,y), \quad z = x + iy \in \mathbb{C},$$

which satisfy the so-called *Cauchy–Riemann equations*

$$u_x - v_y = 0, \quad v_x + u_y = 0. \tag{5.2}$$

Here, the subscript denotes the derivative with respect to the corresponding variable, i.e., $u_x = \partial u/\partial x, \dots, u_{yy} = \partial^2 u/\partial y^2, \dots$). From (5.2) it follows that

$$u_{xx} + u_{yy} = (u_x - v_y)_x + (v_x + u_y)_y = 0, \quad v_{xx} + v_{yy} = 0.$$

Thus, the real and imaginary parts of any analytic function $w(z) = u(x,y) + iv(x,y)$ are harmonic functions.

Consider the following problem for harmonic functions in the half-plane $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$. Let $f \in PC_b(\mathbb{R})$, i.e., (see Definition 3.1.3) f is a bounded and piecewise continuous function of the variable $x \in \mathbb{R}$. We seek a function $u \in C^2(\mathbb{R}^2_+)$ satisfying the Laplace equation

$$u_{xx} + u_{yy} = 0 \quad in \quad \mathbb{R}^2_+ \tag{5.3}$$

and the following *boundary conditions*

$$\lim_{y \to +0} u(x,y) = f(x), \tag{5.4}$$

where x is a point of continuity of the function f.

Problem (5.3)–(5.4) is called the *Dirichlet problem* for the Laplace equation in the half-plane and the function u is called its solution. The Dirichlet problem has many physical interpretations, one of which is given in Remark 6.1.

According to Exercises P.5.12 and P.5.15 below, problem (5.3)–(5.4) has at most one **bounded** solution. We are going to find it. Note that the imaginary part of the analytic function

$$\ln(x + iy) = \ln|x + iy| + i\arg(x + iy), \quad (x,y) \in \mathbb{R}^2_+$$

coincides with $\operatorname{arccot}(x/y) \in]0, \pi[$. Hence, this function is harmonic in $\mathbb{R}^2_+$. Moreover,

$$\lim_{y \to +0} \operatorname{arccot} \frac{x}{y} = \begin{cases} \pi, & if\ x < 0, \\ 0, & if\ x > 0. \end{cases}$$

These properties of the function $\operatorname{arccot}(x/y)$ allow us to use it in order to construct functions harmonic in $\mathbb{R}^2_+$ with piecewise constant boundary values. In particular, the function

$$P_\epsilon(x, y) = \frac{1}{2\pi\epsilon} \left[\operatorname{arccot} \frac{x - \epsilon}{y} - \operatorname{arccot} \frac{x + \epsilon}{y} \right]$$

is harmonic in $\mathbb{R}^2_+$ and satisfies the boundary condition

$$\lim_{y \to +0} P_\epsilon(x, y) = \delta_\epsilon(x) \quad \text{for } |x| \neq \epsilon,$$

where the function $\delta_\epsilon(x)$ is defined in (1.3). On the other hand, if x is a point of continuity for f, then, by virtue of Lemma 2.1,

$$f(x) = \lim_{\epsilon \to 0} \int\limits_{-\infty}^{\infty} \delta_\epsilon(\xi - x) f(\xi) d\xi.$$

This allows us to suppose that the function

$$\mathbb{R}^2 \ni (x, y) \longmapsto \lim_{\epsilon \to 0} \int\limits_{-\infty}^{\infty} P_\epsilon(\xi - x, y) f(\xi) d\xi \tag{5.5}$$

assumes (at the points of continuity of f) the values $f(x)$ as

$y \to +0$ and that this function is harmonic in $\mathbb{R}^2$, since

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\sum_{k=1}^{N} P_\epsilon(\xi_k - x, y) f(\xi_k)(\xi_{k+1} - \xi_k)\right) =$$
$$\sum_{k=1}^{N} f(\xi_k)(\xi_{k+1} - \xi_k))\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) P_\epsilon(\xi_k - x, y)\right] = 0.$$

The formal transition to the limit in (5.5) leads us to the *Poisson integral* (the Poisson formula)

$$u(x,y) = \int_{-\infty}^{\infty} f(\xi) P(x-\xi, y) d\xi, \text{ where } P(x,y) = \frac{1}{\pi}\frac{y}{x^2+y^2}, \quad (5.6)$$

since

$$\lim_{\epsilon\to 0} P_\epsilon(x,y) = -\frac{1}{\pi}\frac{\partial}{\partial x}\left(\operatorname{arccot}\frac{x}{y}\right) = \frac{1}{\pi}\frac{y}{x^2+y^2}.$$

Let us note that condition (5.4) is satisfied by virtue of P.4.1. Moreover, note that the function u is bounded. Indeed,

$$|u(x,y)| = \int_{-\infty}^{\infty} |f(\xi)| P(x-\xi, y)\, d\xi \le C \int_{-\infty}^{\infty} P(x-\xi, y)\, dy = C.$$

We are going to show that the function u is harmonic in $\mathbb{R}^2_+$. Differentiating (5.6), we obtain

$$\frac{\partial^{j+k} u(x,y)}{\partial x^j \partial y^k} = \int_{-\infty}^{\infty} f(\xi) \frac{\partial^{j+k}}{\partial x^j \partial y^k} P(x-\xi, y)\, d\xi \quad \forall j \ge 0,\ \forall k \ge 0. \tag{5.7}$$

Differentiation under the integral sign is possible, because

$$\left|\frac{\partial^{j+k} P(x-\xi, y)}{\partial x^j \partial y^k}\right| \le \frac{C}{1+|\xi|^2} \text{ for } |x| < R,\ \frac{1}{R} < y < R, \tag{5.8}$$

where C depends only on $j \ge 0$, $k \ge 0$ and $R > 1$. From (5.7) it follows that

$$\Delta u(x,y) = \int_{-\infty}^{\infty} f(\xi) \Delta P(x-\xi, y)\, d\xi.$$

However, $\Delta P(x-\xi,y) = \Delta P(x,y) = 0$ in $\mathbb{R}^2_+$, since

$$P(x,y) = -\frac{1}{\pi}\frac{\partial}{\partial x}\left(\operatorname{arccot}\frac{x}{y}\right),\ \Delta\left(\operatorname{arccot}\frac{x}{y}\right) = 0 \text{ and } \Delta\frac{\partial}{\partial x} = \frac{\partial}{\partial x}\Delta.$$

Thus, we have proved that the Poisson integral (5.6) gives a solution of problem (5.3)–(5.4) bounded in $\mathbb{R}^2_+$.

5.3.**P.** *Prove estimate* (5.8).

5.4. Remark. The function P defined in (5.6) is called the *Poisson kernel*. It can be interpreted as the solution of the problem $\Delta P = 0$ in $\mathbb{R}^2_+$, $P(x,0) = \delta(x)$, where $\delta(x)$ is the δ-function[4)].

[4)] The formula (5.6) which gives the solution of the problem $\Delta u = 0$ in $\mathbb{R}^2_+$, $u(x,0) = f(x)$, can be very intuitively interpreted in the following way. The source "stimulating" the physical field $u(x,y)$ is the function $f(x)$ which is the "sum" over ξ of point sources $f(\xi)\delta(x-\xi)$. Since one point source $\delta(x-\xi)$ generates the field $P(x-\xi,y)$, the "sum" of such sources generates (by virtue of linearity of the problem) the field which is the "sum" (i.e., the integral) by ξ of fields of the form $f(\xi)P(x-\xi,y)$. In this case, physicists usually say that we have *superposition* (covering) of fields generated by point sources. This *superposition principle* is observed in many formulae which give solutions of linear problems of mathematical physics (see in this connection formulae (5.10), (6.15), (7.14),...). In these cases, mathematicians usually use the term "*convolution*" (see Section 19).

5.5. Remark. Note that the function P is (in the sense of the definition given above) a solution **unbounded** in $\mathbb{R}^2_+$ of problem (5.3)–(5.4), if $f(x) = 0$ for $x \neq 0$ and $f(0)$ is equal to, for instance, one. On the other hand, for this (piecewise continuous) boundary function f, problem (5.3)–(5.4) admits a bounded solution $u(x,y) \equiv 0$. Thus, there is no uniqueness of the solution of the Dirichlet problem (5.3)–(5.4) in the functional class $C^2(\mathbb{R}^2_+)$. In this connection also see P.5.6, P.5.12 and P.5.15.

5.6. **P.** *Find an unbounded solution* $u \in C^\infty(\mathbb{R}^2_+)$ *of problem* (5.3)–(5.4) *for* $f(x) \equiv 0$.

5.7.**P.** *Let* $k \in \mathbb{R}$, *and* u *be the solution of problem* (5.3)–(5.4) *represented by formula* (5.6). *Find* $\lim\limits_{y\to+0}(x_0+ky,y)$ *in the two cases:*

(1) *f is continuous;*
(2) *f has a jump of the first kind at the point x_0.*

5.8.**P.** *Prove*

5.9. PROPOSITION. *Let Ω and ω be two domains in $\mathbb{R}^2$. Suppose that $u : \Omega \ni (x, y) \longmapsto u(x, y) \in \mathbb{R}$ is a harmonic function and*

$$z(\zeta) = x(\xi, \eta) + iy(\xi, \eta), \quad (\xi, \eta) \in \omega \subset \mathbb{R}^2$$

is an analytic function of the complex variable $\zeta = \xi + i\eta$ with the values in Ω (i.e., $(x, y) \in \Omega$). Then the function

$$U(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta)), \quad (\xi, \eta) \in \omega$$

is harmonic in ω.

5.10.**P.** *Let ρ and φ be polar coordinates in the disk $D = \{\rho < R,\ \varphi \in [0, 2\pi[\}$ of radius R. Suppose that $f \in PC(\partial D)$, i.e., f is a function defined on the boundary ∂D of the disk D and f is continuous everywhere on ∂D except at a finite number of points, where it has discontinuities of the first kind. Consider the Dirichlet problem for the Laplace equation in the disk D: to find a function $u \in C^2(D)$ such that*

$$\Delta u = 0 \text{ in } D, \quad \lim_{\rho \to R} u(\rho, \varphi) = f(R\varphi), \tag{5.9}$$

where $s = R\varphi$ is a point of continuity of the function $f \in PC_b(\partial D)$. Show that the formula

$$u(\rho, \varphi) = \frac{1}{2\pi R} \int_0^{2\pi R} f(s) \frac{(R^2 - \rho^2) ds}{R^2 + \rho^2 - 2R\rho \cos(\varphi - \theta)},\ \theta = \frac{s}{R} \tag{5.10}$$

represents a bounded solution of problem (5.9)*. Formula* (5.10) *was obtained by Poisson in 1823.*

HINT. Make the transformation $w = R\frac{z-i}{z+i}$ of the half-plane $\mathbb{R}^2_+$ onto the disk D and use formula (5.6).

5.11.**P.** *Interpret the kernel of the Poisson integral* (5.10)*, i.e., the function*

$$\frac{1}{2\pi R} \frac{(R^2 - \rho^2)}{R^2 + \rho^2 - 2R\rho \cos(\varphi - \theta)},$$

similar to what has been done in Remark 5.4 with respect to the function P.

5.12.**P.** *Using Theorem 5.13 below, prove the uniqueness of the solution of problem* (5.9) *as well as the uniqueness of the bounded solution of problem* (5.3)–(5.4) *in the assumption that the bounded function f is continuous. (Compare with Remark 5.5.)*

5.13. THEOREM (Maximum principle). *Let Ω be a bounded open set in $\mathbb{R}^n$ with the boundary $\partial\Omega$. Suppose that $u \in C(\bar{\Omega})$ and u is harmonic in Ω. Then u attains its maximum on the boundary of the domain Ω, i.e., there exists a point $x^\circ = (x_1^\circ, \dots, x_n^\circ) \in \partial\Omega$ such that $u(x) \le u(x^\circ)$ $\forall x \in \bar{\Omega}$.*

PROOF. Let $m = \sup\limits_{x \in \partial\Omega} u(x)$, $M = \sup\limits_{x \in \Omega} u(x) = u(x^\circ)$, $x^\circ \in \bar{\Omega}$. Suppose on the contrary that $m < M$. Then $x^\circ \in \Omega$. We set

$$v(x) = u(x) + \frac{M - m}{2d^2}|x - x^\circ|^2,$$

where d is the diameter of the domain Ω. The inequality $|x - x^\circ|^2 \le d^2$ implies that

$$v(x) \le m + \frac{M - m}{2d^2}d^2 = \frac{M + m}{2} < M, \quad x \in \partial\Omega.$$

Note that $v(x^\circ) = u(x^\circ) = M$. Thus, v attains its maximum at a point lying inside Ω. It is know that at such a point $\Delta v \le 0$. Meanwhile,

$$\Delta v = \Delta u + \frac{M - m}{2d^2}\Delta\left(\sum_{k=1}^{n}(x_k - x_k^\circ)^2\right) = \frac{M - m}{2d^2} \cdot 2n > 0.$$

The contradiction obtained proves the theorem. □

5.14.**P.** *In the assumptions of Theorem 5.13, show that u attains its minimum as well on $\partial\Omega$. (This is the reason why the appropriate result (Theorem 5.13) is known as minimum principle.)*

5.15.**P.** *Using Theorem 5.16 below, prove the uniqueness of the bounded solution of problem* (5.9) *as well as problem* (5.3)–(5.4)*. Compare with Exercise P.5.12.*

5.16. THEOREM (on discontinuous majorant). *Let Ω be a bounded open set in $\mathbb{R}^2$ with the boundary $\partial\Omega$, and F a finite set of points $x_k \in \bar{\Omega}$, $k = 1, \dots, N$. Let u and v be two functions harmonic in $\Omega \setminus F$ and continuous in $\bar{\Omega} \setminus F$. Suppose that there exists a constant M such that $|u(x)| \le M$, $|v(x)| \le M$ $\forall x \in \bar{\Omega} \setminus F$. If*

$u(x) \le v(x)$ for any point $x \in \partial\Omega \setminus F$, then $u(x) \le v(x)$ for all points $x \in \bar{\Omega} \setminus F$.

PROOF. First, note that the function $\ln|x|$, where $x \in \mathbb{R}^2 \setminus \{0\}$, is harmonic. We set

$$w_\epsilon(x) = u(x) - v(x) - \sum_{k=1}^{N} \frac{2M}{\ln(d/\epsilon)} \ln \frac{d}{|x - x_k|}.$$

Here, $0 < \epsilon < d$, where d is the diameter of Ω, hence, $\ln(d/|x - x_k|) \ge 0$. Consider the domain Ω_ϵ obtained by cutting off the disks of the radius ϵ centered at the points $x_k \in F$, $k = 1, \dots, N$, from Ω. Obviously, w_ϵ is harmonic in Ω_ϵ, continuous in $\bar{\Omega}_\epsilon$, and $w_\epsilon(x) \le 0$ for $x \in \partial\Omega_\epsilon = \bar{\Omega}_\epsilon \setminus \Omega_\epsilon$. Therefore, by virtue of the maximum principle, $w_\epsilon(x) \le 0$ for $x \in \Omega_\epsilon$. It remains to tend ϵ to zero. □

5.17.**P.** *Let $\Omega \subset \mathbb{R}^2$ be a simply connected open set bounded by a closed Jordan curve $\partial\Omega$. Suppose that a function f is specified on $\partial\Omega$ and is continuous everywhere except a finite number of points at which it has discontinuities of the first kind. Using Proposition 5.9, the Riemann theorem on existence of a conformal mapping from Ω onto the unit disk (see, for instance, [**38**]), prove that there exists a bounded solution $u \in C^2(\Omega)$ of the following Dirichlet problem:*

$$\Delta u = 0 \text{ in } \Omega, \qquad \lim_{x \in \Omega, x \to s} u(x) = f(s), \tag{5.11}$$

where s is a point of continuity of the function $f \in PC(\partial\Omega)$. Using Theorem 5.16, prove the uniqueness of the bounded solution of problem (5.11) *and continuous (make clear in what sense) dependence of the solution on the boundary function f. (Compare with Corollary 22.31.)*

5.18. THEOREM (on the mean value). *Let u be a function harmonic in a disk D of radius R. Suppose that $u \in C(\bar{D})$. Then the value of u at the centre of the disk D is equal to the mean $u(x)$, $x \in \partial D$, i.e., (in notation of P.5.10)*

$$u\big|_{\rho=0} = \frac{1}{2\pi R} \int_0^{2\pi R} u(R, \frac{s}{R}) ds. \tag{5.12}$$

PROOF. The assertion obviously follows from (5.10). □

5.19.**P.** *Let u be a continuous function in a domain $\Omega \subset \mathbb{R}^2$ and u satisfy* (5.12) *for any disk $D \subset \Omega$. Prove (by contradiction) that if $u \neq$ const, then $u(x) < \|u\| \ \forall x \in \Omega$, where $\|u\|$ is the maximum of $|u|$ in $\bar{\Omega}$.*

By virtue of Theorem 5.18, the result of Exercise 5.19 can be formulated in the form of the following assertion.

5.20. Theorem (strong maximum principle). *Let u be a harmonic function in a domain $\Omega \subset \mathbb{R}^2$. If $u \neq$ const, then $u(x) < \|u\| \ \forall x \in \Omega$, where $\|u\|$ is the maximum of $|u|$ in $\bar{\Omega}$.*

5.21.**P.** *In the assumptions of P.5.19, show that u is harmonic in the domain Ω.*

Hint. Let $a \in \Omega$ and $D \subset \Omega$ be a disk centred at a. Suppose that v is a function bounded and harmonic in D such that $v = u$ on ∂D. Using the result of P.5.19, show that the function $w = u - v$ is a constant in D.

5.22. Remark. It follows from P.5.21 and Theorem 5.18 that a function continuous in a domain $\Omega \subset \mathbb{R}^2$ is harmonic if and only if the mean value property (5.12) holds for any disk $D \subset \Omega$. This fact as well as all others in this section is valid for any bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, with a smooth $(n-1)$-dimensional boundary (with an appropriate change of the formulae).

Let us note one more useful fact.

5.23. Lemma (Giraud–Hopf–Oleinik). *Let u be harmonic in Ω and continuous in $\bar{\Omega}$, where the domain $\Omega \Subset \mathbb{R}^n$ has a smooth $(n-1)$-dimensional boundary Γ. Suppose that at a point $x_\circ \in \Gamma$ there exists a normal derivative $\partial u/\partial \nu$, where ν is the outward normal to Γ and $u(x_\circ) > u(x) \ \forall x \in \Omega$. Then $\partial u/\partial \nu\big|_{x=x_\circ} > 0$.*

The proof can be found, for instance, in [**12, 23, 24, 28, 46**].

6. On the heat equation

It is known that in order to heat a body occupying a domain $\Omega \subset \mathbb{R}^3$ from the temperature $u_0 =$ const to the temperature $u_1 =$ const, we must transmit the energy equal to $C \cdot (u_1 - u_0) \cdot |\Omega|$ to the body as the heat, where $|\Omega|$ is the volume of the domain Ω and C is a (positive) coefficient called the specific heat. Let $u(x,t)$ be the temperature at a point $x = (x_1, x_2, x_3) \in \Omega$ at an instant t. We deduce the differential equation which is satisfied by the function u. We assume that the physical model of the real process is such that functions considered in connection with this process (heat energy, temperature, heat flux) are sufficiently smooth. Then the variation of the heat energy in the parallelepiped

$$\Pi = \{x \in \mathbb{R}^3 \mid x_k^\circ < x < x_k^\circ + h_k,\ k = 1, 2, 3\}$$

at time τ (starting from the instant t°) can be represented in the form

$$\begin{aligned} C \cdot [u(x^\circ, t^\circ + \tau) - u(x^\circ, t^\circ)] \cdot |\Pi| + o(\tau \cdot |\Pi|) \\ = C \cdot [u_t(x^\circ, t^\circ) \cdot \tau + o(\tau)] \cdot |\Pi| + o(\tau \cdot |\Pi|), \end{aligned} \tag{6.1}$$

where $|\Pi| = h_1 \cdot h_2 \cdot h_3$ and $o(A)$ is small o of $A \in \mathbb{R}$ as $A \to 0$.

This variation of the heat energy is connected with the presence of a heat flux through the boundary of the parallelepiped Π. According to the Fourier law, the heat flux per unit of time through an area element in direction of the normal to this element is proportional with a (negative) coefficient of proportionality to $-k$ derivative of the temperature along this normal. The coefficient of proportionality $k > 0$ is called the coefficient of heat conductivity. Thus, the quantity of the energy entered into the parallelepiped Π during the time τ through the area element $x_1 = x_1^\circ + h_1$ is equal to

$$k(x_1^\circ + h_1, x_2^\circ, x_3^\circ) \cdot \frac{\partial u}{\partial x_1}(x_1^\circ + h_1, x_2^\circ, x_3^\circ; t^\circ) \cdot \tau \cdot h_2 \cdot h_3 + o(\tau \cdot |\Pi|),$$

and gone out during the same time through the area element $x_1 = x_1^\circ$ is equal to

$$k(x_1^\circ, x_2^\circ, x_3^\circ) \cdot \frac{\partial u}{\partial x_1}(x_1^\circ, x_2^\circ, x_3^\circ; t^\circ) \cdot \tau \cdot h_2 \cdot h_3 + o(\tau \cdot |\Pi|).$$

Therefore, the variation of the heat energy in Π caused by the heat flux along the axis x_1 is equal to

$$\left[\frac{\partial}{\partial x_1}(k(x^\circ)\frac{\partial u}{\partial x_1}(x^\circ, t^\circ))h_1 + o(h_1)\right] \tau \cdot h_2 \cdot h_3 + o(\tau \cdot |\Pi|).$$

It is clear that the variation of the heat energy in Π in all three directions is equal to the total variation of the heat energy in Π, i.e., (6.1). By dividing the equality obtained in this way by $\tau \cdot |\Pi|$ and tending τ, h_1, h_2, and h_3 to zero, we obtain the heat equation

$$C\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_1}\left(k\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(k\frac{\partial u}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(k\frac{\partial u}{\partial x_3}\right). \tag{6.2}$$

If the coefficients C and k are constant, then equation (6.2) can be rewritten in the form

$$\frac{\partial u}{\partial t} = a\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}\right), \quad \text{where } a = \frac{k}{C} > 0.$$

6.1. Remark. In the case when the distribution of the temperature does not depend on the time, i.e., $u_t = 0$, the temperature u satisfies the Laplace equation (if $k = \text{const}$). Thus, the Dirichlet problem for the Laplace equation (see Section 5) can be interpreted as a problem on the distribution of stabilized (stationary) temperature in the body, if the distribution of the temperature on the surface of the body is known.

If we are interested in distribution of the temperature inside the body, where (during some time) influence of the boundary conditions is not very essential, then we idealize the situation and consider the following problem:

$$\begin{aligned} C\frac{\partial u}{\partial t} &= \operatorname{div}(k \cdot \operatorname{grad} u), \quad (x, y, z) \in \mathbb{R}^3, \quad t > 0, \\ u\big|_{t=0} &= f(x, y, z), \end{aligned}$$

where f is the distribution of the temperature (in the body without boundary, i.e., in $\mathbb{R}^3$) at the instant $t = 0$. This problem is sometimes called by the Cauchy problem for the heat equation.

Suppose that f and k, hence, also u do not depend on y and z. Then u is a solution of the problem

$$C\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k \cdot \frac{\partial u}{\partial x}\right), \quad (x, t) \in \mathbb{R}^2_+ = \{x \in \mathbb{R},\ t > 0\}, \tag{6.3}$$

$$u\big|_{t=0} = f(x), \tag{6.4}$$

6.2. HINT. *The method which was used to solve problem* (5.3)–(5.4)[1)] *suggests that the solution to problem* (6.3)–(6.4) *can be represented by the formula*

$$u(x,t) = \int_{-\infty}^{\infty} f(\xi) v(x-\xi, t)\, d\xi,$$

where v *is the solution of equation* (6.3)*, satisfying the condition*

$$\lim_{t\to+0} v(x,t) = \delta(x), \quad \textit{where } \delta \textit{ is the } \delta\textit{-function.} \tag{6.5}$$

[1)] See note 4 in Section 5.

Below (see Theorem 6.5) we show that this is true.

Let us try to find the function v. It satisfies the following conditions:

$$C\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left(k \cdot \frac{\partial v}{\partial x}\right), \quad \int_{-\infty}^{\infty} Cv\, dx = Q, \tag{6.6}$$

where Q is the total quantity of the heat that in our case is equal to C. Thus, we see that v is a function G of five independent variables x, t, C, k, and Q, i.e.,

$$v = G(x, t, C, k, Q). \tag{6.7}$$

6.3. REMARK. The method with the help of which we seek for the function v is originated from mechanics [**55**]. It is known as the dimensionless parameters (variables) method.

Note that the units of measurement of the quantities v, x, t, and Q in the SI system, for instance, are the following: $[v] = \mathrm{K}$, $[x] = \mathrm{m}$, $[t] = \mathrm{sec}$, $[Q] = \mathrm{w}$. By virtue of (6.6), $[C][v]/[t] = [k][v]/[x]^2$, $[C][v][x] = [Q]$; therefore the dimensions of the quantities C and k are expressed by the formulae:

$$[C] = \mathrm{w}/(\mathrm{m}\cdot\mathrm{K}), \quad [k] = \mathrm{w}\cdot\mathrm{m}/(\mathrm{sec}\cdot\mathrm{K}).$$

Since C, k, and Q play the role of the parameters of the function $v(x,t)$, it is preferable to express the units of measurement v and, say, x via $[t]$, $[C]$, $[k]$ and $[Q]$. We have

$$[x] = \sqrt{[t]\cdot[k]/[C]}, \quad [v] = [Q]/\sqrt{[t]\cdot[k]\cdot[C]}.$$

Let us take another system of units of measurement

$$[t^*] = \sigma_t[t],\ [C^*] = \sigma_C[C],\ [k^*] = \sigma_k[k],\ [Q^*] = \sigma_Q[Q],$$

where σ_t, σ_C, σ_k, σ_Q are scaling coefficients, i.e., positive (dimensionless) numbers. We formulate the question: what are the values of the scaling coefficients σ_x and σ_v (for the variables x and v which are "derivatives" of the chosen "basic" physical variables t, C, k, and Q)? We have

$$\begin{aligned}[x^*] = \sqrt{\frac{[t^*][k^*]}{[C^*]}} = \sigma_x[x] = \sigma_x\sqrt{[t][k]/[C]}& \\ = \sigma_x\sqrt{\sigma_C[t^*][k^*]/(\sigma_t\sigma_k[C^*])}&.\end{aligned}$$

Therefore,

$$\sigma_x = \sqrt{\sigma_t\sigma_k/\sigma_C} \quad\text{and similarly}\quad \sigma_v = \sigma_Q/\sqrt{\sigma_t\sigma_k\sigma_C}. \tag{6.8}$$

The numerical values $t^*, \ldots, v^*$ of the variables $t, \ldots, v$ in the new system of units are determined from the relation

$$t^*[t^*] = t[t], \ldots, v^*[v^*] = v[v].$$

Thus,

$$\begin{gathered}t^* = \frac{t}{\sigma_t}, \quad C^* = \frac{C}{\sigma_C}, \quad k^* = \frac{k}{\sigma_k}, \quad Q^* = \frac{Q}{\sigma_Q}, \\ x^* = x\sqrt{\frac{\sigma_C}{\sigma_t\sigma_k}}, \quad v^* = v\frac{\sqrt{\sigma_t\sigma_k\sigma_C}}{\sigma_Q}.\end{gathered}$$

For instance, if $[t] = \text{sec}$ and $[t^*] = \text{hour}$, then $\sigma_t = 3600$ and $t^* = t/3600$.

Let us now note that relation (6.7) expresses a law that does not depend on the choice of the units. Therefore,

$$v^* = G(x^*, t^*, C^*, k^*, Q^*) \tag{6.9}$$

with the same function G. Now, we choose the system of units such that $t^* = C^* = k^* = Q^* = 1$, i.e., we set $\sigma_t = t$, $\sigma_C = C$, $\sigma_k = k$, $\sigma_Q = Q$. Then

$$x^* = x\sqrt{C/(kt)}, \quad v^* = v\sqrt{tkC}/Q.$$

Hence, by virtue of (6.9), we have

$$v(x,t) = \frac{Q}{\sqrt{kCt}} g\left(\sqrt{\frac{C}{kt}} \cdot x\right), \quad \text{where } g(y) = G(y,1,1,1,1). \tag{6.10}$$

6.4. Remark. We can come to formula (6.10) strongly mathematically. Namely, by making the change of variables

$$t^* = \frac{t}{\sigma_t},\ C^* = \frac{C}{\sigma_C},\ k^* = \frac{k}{\sigma_k},\ Q^* = \frac{Q}{\sigma_Q},\ x^* = \frac{x}{\sigma_x},\ v^* = \frac{v}{\sigma_v},$$

we require v^* be equal to $G(x^*, t^*, C^*, k^*, Q^*)$, i.e., we require that

$$C^* \cdot \frac{\partial v^*}{\partial t^*} = \frac{\partial}{\partial x^*}\left(k^* \cdot \frac{\partial v^*}{\partial x^*}\right), \qquad \int\limits_{-\infty}^{\infty} C^* v^* dx = Q^*.$$

Then (6.6) necessarily implies (6.8). Choosing, as before, the scaling coefficients σ_t, σ_C, σ_k, and σ_Q, we again obtain (6.10).

Nevertheless, it is helpful to use the dimension arguments. Firstly, they allow us to test the correctness of involving some parameters when formulating the problem: both sides of any equality used in the problem should have consistent dimensions. Secondly, the reasoning of dimension allows to find the necessary change of variables (not necessarily connected only with scaling coefficients). All these facts allow automatically (hence, easily) to get rid of "redundant" parameters and so to simplify the analysis as well as the calculations[2)]. Moreover, the passage to dimensionless coordinates allows us to apply the reasoning by similarity that sometimes essentially simplify the solution of rather difficult problems (see [**55**]).

[2)] Consider the problem on the temperature field of an infinite plate of a thickness $2S$, with an initial temperature $T_0 = \text{const}$, in the case, when there is the heat transfer on the surface of the plate (with the coefficient of the heat transfer α) with the medium whose temperature is equal to $T_1 = \text{const}$. In other words, we consider the problem

$$\frac{\partial T}{\partial \tau} = a\frac{\partial^2 T}{\partial \xi^2},\ \tau > 0,\ |\xi| < S;\ \mp k\frac{\partial T}{\partial \xi}\Big|_{\xi=\pm S} = \alpha(T - T_1)\Big|_{\xi=\pm S};\ T\Big|_{\tau=0} = T_0.$$

The function $T = f(\tau, \xi, a, S, k, \alpha, T_1, T_0)$ depends *a priori* on eight parameters. The tabulating the values of such a function, if each of the parameters run over at least ten values, is unreasonable, because one should analyze million pages. At the same time, the passage to the dimensionless parameters

$$u = (T - T_1)/(T_1 - T_0),\ x = \xi/S,\ t = a\tau/S^2,\ \sigma = k/\alpha S$$

reduces this problem to the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ t > 0,\ |x| < 1;\ \left(u \pm \sigma\frac{\partial u}{\partial x}\right)\Big|_{x=\pm 1} = 0;\ u\big|_{t=0} = 1, \quad (6.11)$$

whose solution $u = u(t, x, \sigma)$ can be represented (this is very important for applications) in the form of compact tables (one page for each value of $\sigma \geq 0$).

In order to find the function g and, hence, v we substitute expression (6.10) into the heat equation (6.2). We obtain

$$Q\sqrt{C/kt^3}[g(y)/2+y\cdot g'(y)/2+g''(y)] = 0,\ \text{ i.e., }\ (yg(y))'/2+g''(y) = 0.$$

Thus, the function g satisfies the linear equation

$$g'(y) + yg(y)/2 = \text{const}\,. \qquad (6.12)$$

If g is even, i.e., $g(-y) = g(y)$, then $g'(0) = 0$; therefore, the function g satisfies the homogeneous equation (6.12) whose solution, obviously, is represented by the formula $g(y) = A \cdot \exp(-y^2/4)$. The constant A is to be determined from the second condition in (6.6):

$$Q = \int_{-\infty}^{\infty} Cv dx = ACQ/\sqrt{kCt}\int_{-\infty}^{\infty} e^{-Cx^2/4kt}dx = 2AQ\int_{-\infty}^{\infty} e^{-\xi^2}\,d\xi,$$

i.e., (accounting formula (1.6)) $A = 1/(2\sqrt{\pi})$, hence,

$$v(x,t) = (Q/2\sqrt{kC\pi t}) \cdot \exp(-Cx^2/4kt). \qquad (6.13)$$

6.5. Theorem. *Let $f \in C(\mathbb{R})$, and for some $\sigma \in [1, 2[$, $a > 0$, and $M > 0$ the following inequality holds:*

$$|f(x)| \leq M\exp(a|x|^\sigma) \quad \forall x \in \mathbb{R}. \qquad (6.14)$$

Then the function $u : \mathbb{R}^2_+ = \{(x,t) \in \mathbb{R}^2 \mid t > 0\} \to \mathbb{R}$ defined by the formula

$$u(x,t) = \int_{-\infty}^{\infty} f(\xi)P(x - \xi, t)d\xi,\ P(x,t) = (1/2\sqrt{\pi t})\exp(-x^2/4t), \qquad (6.15)$$

is a solution of the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad in \quad \mathbb{R}^2_+ = \{(x,t) \in \mathbb{R}^2 \;\mid\; t > 0\}. \tag{6.16}$$

This solution is infinitely differentiable and satisfies the initial condition (which is sometimes called the Cauchy condition)

$$\lim_{t \to +0} u(x,t) = f(x). \tag{6.17}$$

Moreover, $\forall T > 0\ \exists C(T) > 0$ *such that*

$$|u(x,t)| \le C(T)\exp((2a|x|)^\sigma) \quad \forall x \in \mathbb{R} \quad and \quad \forall t \in [0,T]. \tag{6.18}$$

PROOF. From construction of function (6.13), it follows that the function $P(x,t) = (4\pi t)^{-1/2}\exp(-x^2/4t)$ satisfies relations (6.6) in which $C = k = Q = 1$. Therefore, formula (6.17) follows from P.4.2, and (6.16) as well as the smoothness of the function u follows from the known theorem on differentiability of integrals with respect to the parameter (see, for instance, [**72**]), because the appropriate integral converges uniformly, since $\forall R > 1\ \forall \epsilon > 0\ \exists N > 1$ such that

$$\int\limits_{|\xi|>N} \left| \frac{\partial^{j+k}}{\partial x^j \partial t^k}(f(\xi)P(x-\xi,t)) \right| d\xi < \epsilon \tag{6.19}$$

for $x \in [-R,R]$, $t \in [1/R,R]$. For $j+k > 0$, the integrand in (6.19) can be estimated for the specified x and t via $C_R f(\xi)P(x-\xi,t)$. Therefore, in order to prove inequality (6.19) it is sufficient to establish estimate (6.18). Note that for $\sigma \in [1,2[$,

$$|\xi|^\sigma \le 2^\sigma(|x|^\sigma + |\xi - x|^\sigma), \quad |\xi - x|^\sigma \le \epsilon(\xi - x)^2 + C_\epsilon|\xi - x|.$$

Take ϵ such that $1 - 4T \cdot a_\circ \cdot \epsilon > 0$, where $a_\circ = a \cdot 2^\sigma$. Then for $t \le T$, (compare with [**25**])

$$\begin{aligned}
|u(x,t)| &\le \frac{M}{2\sqrt{\pi}} \int e^{a|\xi|^\sigma} \cdot \exp(-(x-\xi)^2/4t)\frac{d\xi}{\sqrt{t}} \\
&\le M_1 e^{(2a|x|)^\sigma} \int e^{a(2|\xi - x|)^\sigma} \cdot e^{-((x-\xi)/2\sqrt{t})^2}\frac{d\xi}{2\sqrt{t}} \\
&\le M_1 e^{(2a|x|)^\sigma} \int e^{-(1-4Ta_\circ\epsilon)(x-\xi)^2/4t} \cdot e^{a_\circ C_\xi(|x-\xi|/2\sqrt{t})2\sqrt{t}}\frac{d\xi}{2\sqrt{t}}.
\end{aligned}$$

Setting $\eta = (\xi - x)(1 - 4Ta_\circ\epsilon)^{1/2}/(2\sqrt{t})$, we have

$$|u(x,t)| \le C(T)e^{(2a|x|)^\sigma} \int e^{-\eta^2 + \alpha|\eta|\cdot 2\sqrt{t}}d\eta.$$

This implies estimate (6.18), if we note that

$$\int_0^\infty e^{-\eta^2+\alpha\eta\cdot 2\sqrt{t}}d\eta = e^{\alpha^2 t}\int_0^\infty e^{-(\eta-\alpha\sqrt{t})}d\eta \le e^{\alpha^2 t}\int_0^\infty e^{-\zeta^2}d\zeta.$$

□

6.6. Remark. In general, there exists a solution of problem (6.16)–(6.17) different from (6.15). For instance, the function $u(x,t)$ represented by the series

$$u(x,t) = \sum_{m=0}^{\infty} \varphi^{(m)}(t)\cdot x^{2m}/(2m)!, \quad (x,t)\in\mathbb{R}^2_+, \tag{6.20}$$

in which the function $\varphi\in C^\infty(\mathbb{R})$ satisfies the conditions:

$$\operatorname{supp}\varphi\subset[0,1],\ \forall m\in\mathbb{Z}_+\ |\varphi^{(m)}(t)|\le(\gamma m)!,\ \text{where } 1<\gamma<2, \tag{6.21}$$

is, obviously, a solution of problem (6.16)–(6.17) for $f=0$. (The condition $\gamma<2$ is needed for uniform (with respect to x and t, $|x|\le R<\infty$) convergence of series (6.20) ad its derivatives.) This simple but important fact was observed in 1935 by A.N. Tikhonov [**64**], who, while constructing series (6.20), used the result by Carleman [**9**] on the existence of a non-zero function φ with properties (6.21). It is important to emphasize that the non-zero solution (6.20) of the heat equation constructed by Tikhonov (satisfying the condition $u(x,0)=0$) grows for $|x|\to\infty$ faster than $\exp Cx^2$ $\forall C>0$ (and slower than $\exp Cx^\sigma$, where $\sigma=2/(2-\gamma)>2$). On the other hand, one can show, by using the maximum principle for the heat equation (see, for instance, [**20, 25, 44, 64**]) that the solution of problem (6.16)–(6.17) is unique, if condition (6.18) holds. The uniqueness theorem for a more large class of function was proved in 1924 by Holmgren [**27**].

From Remark 6.6, it follows

6.7. Theorem. *Let $f\in C(\mathbb{R})$, and f satisfy* (6.14). *Then formula* (6.15) *represents a solution of problem* (6.16)–(6.17) *and this solution is unique in class* (6.18).

7. The Ostrogradsky–Gauss formula. The Green formulae and the Green function

Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth $(n-1)$-dimensional boundary $\partial\Omega$. Let $f = (f_1, \dots, f_n)$ be a vector-function such that $f_k \in C(\bar{\Omega})$ and $\partial f_k/\partial x_k \in PC(\Omega)$ $\forall k$. It is known [**63, 72**] that in this case the *Ostrogradsky–Gauss formula*[1)]

$$\int\limits_{\Omega} \sum_{k=1}^{n} \frac{\partial f_k(x)}{\partial x_k}\, dx = \int\limits_{\partial\Omega} \sum_{k=1}^{n} f_k(x) \cdot \alpha_k \, d\Gamma \tag{7.1}$$

holds, where $\alpha_k = \alpha_k(x)$ is the cosine of the angle between the outward normal ν to $\Gamma = \partial\Omega$ at the point $x \in \Gamma$ and kth coordinate axis, and $d\Gamma$ is the "area element" of Γ. For $n = 1$, formula (7.1) becomes the Newton–Leibniz formula.

[1)] Formula (7.1) is a special case of the important Stokes theorem on integration of differential forms on manifolds with boundary (see, for instance, [**63, 72**]), which can be represented by the Poincaré formula: $\int_\Omega d\omega = \int_{\partial\Omega} \omega$. The Poincaré formula implies (7.1) for

$$\omega = \sum_k f_k(x) dx_1 \bigwedge \dots \bigwedge dx_{k-1} \bigwedge dx_{k+1} \bigwedge \dots \bigwedge dx_n,$$

because

$$d\omega = \sum (\partial f_k(x)/\partial x_k) dx, \text{ and } \omega\big|_{\partial\Omega} = \sum f_k(x) \alpha_k d\Gamma.$$

If $f_k(x) = A_k(x) v(x)$, where $v \in PC^2(\Omega) \cap C^1(\bar{\Omega})$, then (7.1) implies that

$$\int\limits_{\Omega} v \cdot \left(\sum_{k=1}^{n} \frac{\partial A_k}{\partial x_k} \right) dx = -\int\limits_{\Omega} \sum_{k=1}^{n} A_k \frac{\partial v}{\partial x_k} dx + \int\limits_{\partial\Omega} \sum_{k=1}^{n} A_k \cdot v \cdot \alpha_k d\Gamma. \tag{7.2}$$

Setting $A_k = \frac{\partial u}{\partial x_k}$, where $u \in PC^2(\Omega) \cap C^1(\bar{\Omega})$, we obtain the *first Green formula*

$$\int\limits_{\Omega} v \cdot \Delta u dx = \int\limits_{\partial\Omega} v \frac{\partial u}{\partial \nu} d\Gamma - \int\limits_{\Omega} \sum_{k=1}^{n} \frac{\partial u}{\partial x_k} \cdot \frac{\partial v}{\partial x_k} dx, \tag{7.3}$$

where Δ is the Laplace operator (see Section 5). Renaming u by v in (7.3) and v by u and subtracting the formula obtained from (7.3), we obtain the so-called *second Green formula* for the Laplace operator

$$\int\limits_{\Omega} (v \cdot \Delta u - u \cdot \Delta v) dx = \int\limits_{\partial\Omega} \left(v \cdot \frac{\partial u}{\partial \nu} - u \cdot \frac{\partial v}{\partial \nu} \right) d\Gamma. \tag{7.4}$$

Formula (7.4) implies (if we set $v \equiv 1$) the remarkable corollary:

$$\int\limits_{\Omega} \Delta u \, dx = \int\limits_{\partial\Omega} \frac{\partial u}{\partial \nu} d\Gamma. \tag{7.5}$$

In particular, it the function $u \in C^1(\bar{\Omega})$ is harmonic in Ω, then $\int\limits_{\partial\Omega} \frac{\partial u}{\partial \nu} d\Gamma = 0$. This is the so-called *integral Gauss formula*.

Rewrite formula (7.4) in the form

$$\int\limits_{\Omega} u(y) \Delta v(y) dy = \int\limits_{\Omega} v(y) \Delta u(y) \, dy + \int\limits_{\partial\Omega} \left[u(y) \frac{\partial v}{\partial \nu}(y) - v(y) \frac{\partial u}{\partial \nu}(u) \right] d\Gamma. \tag{7.6}$$

Let us take a point $x \in \Omega$. Replace the function u in (7.6) by the function $E_\alpha(x, \cdot) \in PC^2(\Omega)$ which depends on x as an parameter and which satisfies the equation

$$\Delta_y E_\alpha(x, y) \equiv \sum_{k=1}^{n} \frac{\partial^2}{\partial y_k^2} E_\alpha(x, y) = \delta_\alpha(x - y), \tag{7.7}$$

where δ_α is defined in (1.3), and $1/\alpha \gg 1$. Using the result of Exercise 7.1, we tend α to zero. As a result, taking into account Lemma 2.1, we obtain

$$u(x) = \int\limits_{\Omega} E(x - y) \Delta u(y) dy + \int\limits_{\partial\Omega} \left[u(y) \frac{\partial E(x - y)}{\partial \nu} - E(x - y) \frac{\partial u(y)}{\partial \nu} \right] dy. \tag{7.8}$$

7.1. **P.** *Using the result of P.5.2 and Theorem 5.13, show that the general solution of equation* (7.7) *depending only on* $|x - y|$ *can be represented in the form* $E_\alpha(x - y) + \mathrm{const}$, *where the function*

$E_\alpha \in C^1(\mathbb{R}^n)$ coincides for $|x| \geq \alpha$ with the function

$$E(x) = \begin{cases} (1/2\pi) \cdot \ln|x| & x \neq 0,\ n = 2, \\ -1/((n-2)\sigma_n \cdot |x|^{n-2}) & x \neq 0,\ n \geq 3, \end{cases} \tag{7.9}$$

and for $|x| < \alpha$, the estimate $|E_\alpha(x)| \leq |E(x)|$ holds. Here, σ_n denotes (see P.1.1) the area of the unit sphere in $\mathbb{R}^n$.

Hint. By virtue of (7.5) and (1.3),

$$\int\limits_{|x|=\alpha} (\partial E_\alpha/\partial\nu) d\Gamma = \int\limits_{|x|<\alpha} \Delta E_\alpha dx = 1.$$

Let $x \in \Omega$. We take[2)] the function $g(x,\cdot) : \bar{\Omega} \ni y \longmapsto g(x,y)$, which is the solution of the following Dirichlet problem for the homogeneous Laplace equation with the special boundary condition

$$\Delta_y g(x,y) = 0 \ \text{ in } \ \Omega, \quad g(x,y) = -E(x-y) \ \text{ for } \ y \in \Gamma. \tag{7.10}$$

Substituting the function $g(x,\cdot)$ into formula (7.6) for the function v and summing termwise the equality obtained with (7.8), as a result, we have the following integral representation of the function $u \in PC^2(\Omega) \cap C^1(\bar{\Omega})$:

$$u(x) = \int\limits_{\Omega} G(x,y)\Delta u(y)\, dy + \int\limits_{\partial\Omega} \frac{\partial G(x,y)}{\partial\nu} u(y)\, d\Gamma, \tag{7.11}$$

where

$$G(x,y) = E(x-y) + g(x,y). \tag{7.12}$$

[2)] In Section 22 the theorem is presented on existence of solutions for problems much more general than problem (7.10). In the same section the theorem concerning the smoothness of the solutions is given.

Function (7.12) is called the *Green function* of the Dirichlet problem for the Laplace equation

$$\Delta u = f \ \text{ in } \ \Omega, \quad u = \varphi \ \text{ on } \ \partial\Omega. \tag{7.13}$$

This term is connected with the fact that, by virtue of (7.11), the solution of problem (7.13), where $f \in PC(\Omega)$, $\varphi \in C(\partial\Omega)$, can be represented , using the function G, in the form

$$u(x) = \int\limits_{\Omega} f(y)G(x,y)\,dy + \int\limits_{\partial\Omega} \varphi(y)\frac{\partial G(x,y)}{\partial \nu}\,d\Gamma. \tag{7.14}$$

Formula (7.14) is often called the *Green formula.*

7.2. **P.** *Let* $\Omega = \mathbb{R}^n_+$, *where* $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n \mid x' \in \mathbb{R}^{n-1},\ x_n > 0\}$ *and* $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. *Show that in this case* $G(x,y) = E(x,y) - E(x^*, y)$, *where* $x^* = (x', -x_n)$ *is the flip of the point* x *over the hyperplane* $x_n = 0$. *Verify (compare with* (5.6)*) that*

$$\frac{\partial G(x,y)}{\partial y}\Big|_{y_n=0} = \frac{2}{\sigma_n}\frac{x_n}{[(x_1-y_1)^2 + \cdots + (x_{n-1}-y_{n-1})^2 + x_n]^{n/2}}.$$

8. The Lebesgue integral[1)]

[1)] The theory of the Lebesgue integral is, as a rule, included now in the educational programs for the students of low courses. Nevertheless, maybe some readers are not familiar with this subject. This section and the following one are addressed to these readers. At the first reading, one can, having a look at the definitions and formulations of the assertions of this sections, go further. For what follows, it is important to know at least two facts:

(1) if a function f is piecewise continuous in $\Omega \Subset \mathbb{R}^n$, then f is integrable in the sense of Riemann;
(2) a function integrable in the sense of Riemann is integrable in the sense of Lebesgue, and its Riemann integral coincides with its Lebesgue integral (see P.8.15 below).

If the need arises (when we consider the passage to the limit under the integral sign, the change of the order of integration and so on), it is advisable to come back to more attentive reading of Sections 8–9 and the text-books cited.

In Sections 1–2 we have outlined the idea of representability (in other words, of determination) of a function by its "average values". This idea is connected with the notion of an integral. Let us recall that the definition of an integral, known to all the students since their first year of studies, has been given by Cauchy. It was the first analytic definition of the integral. Cauchy gave the first strict definition of continuity of a function. He proved that functions continuous on a closed interval are integrable. In connection with the development (Dirichlet, Riemann) of the concept of a function as a pointwise mapping into the real axis, the question arose concerning the class of functions for which the integral in the sense of Cauchy exists. This question has been answered by Riemann (see, for instance, [**65**]). This is the reason, why the integral introduced by Cauchy is called the Riemann integral.

The space of functions integrable in the Riemann sense is rather large. However, it is not complete (see note 5 in Section 8) with respect to the convergence defined by the Riemann integral similarly to the fact the set of rational numbers (in contrast to the set of real numbers) is not complete with respect to the convergence defined by the Euclidian distance on the line. Actually, let

$$f_n(x) = x^{-1/2} \quad \text{for} \quad x \in]1/n, 1] \quad \text{and} \quad f_n(x) = 0 \quad \text{for} \quad x \in]0, 1/n].$$

Obviously, $\int_0^1 |f_m(x) - f_n(x)|\, dx \to 0$ as m and $n \to \infty$, i.e., the sequence $\{f_k\}$ is a fundamental sequence (see note 5 in Section 8) with respect to the convergence defined by the riemann integral. It follows from the definition the Riemann integral that the function $x \longmapsto x^{-1/2}$ (to which the sequence $\{f_n(x)\}$ converges pointwise) is not integrable in the Riemann sense. Moreover, one can easily show, using P.8.15 and Theorem 8.17, that there exists no function f integrable in the Riemann sense such that $\lim\limits_{m\to\infty} \int |f(x) - f_m(x)|\, dx = 0$, i.e., the space of functions integrable in the Riemann sense, is not complete with respect to the convergence defined by the Riemann integral (see also Exercise P.8.23).

This reason as well as some others stimulated (see, for instance, [**65**]) the development of the notion of integral. A particular role due to its significance is played by the Lebesgue integral. In 1901 26-year Lebesgue introduced (see Definition 8.12 below) the space $L(\Omega)$ of functions defined on an open set $\Omega \subset \mathbb{R}^n$ and called now by integrable in the Lebesgue sense and the integral that is called now by his name (see Definition 8.12). This integral was defined by Lebesgue as the functional

$$\int : L(\Omega) \ni f \longmapsto \int f \in \mathbb{R},$$

which in the case $\Omega =]a, b[$ is noted by the standard symbol and possesses the following six properties:

(1) $\int\limits_a^b f(x)\,dx = \int\limits_{a+h}^{b+h} f(x-h)\,dx$ for any a, b, and h.

(2) $\int\limits_a^b f(x)\,dx + \int\limits_b^c f(x)\,dx + \int\limits_c^a f(x)\,dx = 0$ for any a, b, and c.

(3) $\int\limits_a^b [f(x)+g(x)]\,dx = \int\limits_a^b f(x)d\,x + \int\limits_a^b g(x)\,dx$ for any a and b.

(4) $\int\limits_a^b f(x)\,dx \geq 0$, if $f \geq 0$ and $b > a$.

(5) $\int\limits_0^1 1 \cdot dx = 1$.

(6) If, when n increases, the function $f_n(x)$ tends increasing to $f(x)$, then the integral of $f_n(x)$ tends to the integral of $f(x)$.

"Condition 6, – wrote Lebesgue [**39**], – takes a special place. It have neither the character of simplicity as the first five nor the character of necessity". Nevertheless, it is Condition 6 that became a corner-stone in Lebesgue's presentation of his theory of integration.

Below we give[2)] the construction theory of the Lebesgue integral and the space $L(\Omega)$ and formulate results of the theory of the Lebesgue integral that we need in further considerations.

[2)] Following mainly the book by G.E. Shilov and B.L. Gurevich [**58**]; also see [**36, 56**].

8.1. Definition. A set $A \subset \Omega$ is said to be a *set of zero measure*, if $\forall \epsilon > 0$ there exists a family of parallelepipeds

$$\Pi_k = \{x = (x_1, \dots, x^n) \in \mathbb{R}^n \mid x_j \in]a_{j_k}, b_{j_k}[\}, \quad k \in \mathbb{N},$$

such that

(1) $A \subset \bigcup\limits_{k=1}^{\infty} \Pi_k$;

(2) $\sum\limits_{k=1}^{\infty} \mu(\Pi_k) < \epsilon$, where $\mu(\Pi_k)$ is the measure of the parallelepiped, i.e., $\mu(\Pi_k) = \prod\limits_{j=1}^{\infty} (b_{j_k} - a_{j_k})$.

8.2. Definition. We say that a property $P(x)$ depending on the point $x \in \Omega$ is valid *almost everywhere*, if the set of points x, at which $P(x)$ does not hold, is of zero measure. We also say that $P(x)$ is valid *for almost all* $x \in \Omega$.

8.3. Definition. By a *step function* in Ω we mean a function $f : \Omega \to \mathbb{R}$ that is a finite linear combination of characteristic functions of some parallelepipeds Π_k, $k = 1, \dots, N$, $N \in \mathbb{N}$, i.e.,

$$f(x) = \sum_{k=1}^{N} c_k \cdot 1_{\Pi_k}(x), \quad c_k \in \mathbb{R},\ x \in \Omega. \tag{8.1}$$

In this case the sum $\sum\limits_{k=1}^{N} c_k \cdot \mu(\Pi_k)$, denoted by $\int f$, is called the integral of the step function (8.1).

8.4. Definition. A function $f : \Omega \ni x \longmapsto f(x) \in \mathbb{C}$ with complex values finite for almost all $x \in \Omega$ is called *measurable*, if there exists sequences $\{g_m\}$ and $\{h_m\}$ step functions in Ω such that $\lim\limits_{m\to\infty} [g_m(x) + i h_m(x)] = f(x)$ for almost all $x \in \Omega$.

8.5.**P.** *Show that if f and g are measurable in Ω and $h : \mathbb{C}^2 \to \mathbb{C}$ is continuous, then the function $\Omega \ni x \longmapsto h(f(x), g(x)) \in \mathbb{C}$ is also measurable.*

8.6. Definition. A set $A \subset \Omega$ is called *measurable*, if 1_A is a measurable function.

8.7.**P.** *Show that any open set and any closed set are measurable. Show that the complement of a measurable set is measurable. Show that a countable union and a countable intersection of measurable sets are measurable.*

8.8. Definition. We say that f belongs to the *class* L^+ (more exactly, to $L^+(\Omega)$), if in Ω there exists an increasing sequence $\{h_k\}_{k=1}^{\infty}$ of step functions such that $\int h_m \le C\ \forall m$ for a constant C and, moreover, $h_m \uparrow f$. The last condition means[3)] that $h_1(x) \le h_2(x) \le \dots \le h_m(x) \le \dots$ and $\lim\limits_{m\to\infty} h_m(x) = f(x)$ for almost all $x \in \Omega$.

[3] The notation $h_m \downarrow f$ has the similar meaning.

One can readily prove

8.9. PROPOSITION. *If $f \in L^+(\Omega)$, then f is measurable in Ω.*

8.10. DEFINITION. The (Lebesgue) *integral* of a function $f \in L^+(\Omega)$ is defined by the formula $\int f = \lim_{m\to\infty} \int h_m$, where $\{h_m\}_{m=1}^\infty$ is an increasing sequence of functions determining f (see Definition 8.8).

One can show that Definition 8.10 is correct, i.e., $\int f$ depends only on f but not on the choice of the sequence $h_m \uparrow f$.

8.11. LEMMA (see, for instance, [**56**]). *Let $f_n \in L^+$, $f_n \uparrow f$, $\int f_n \le C\ \forall n$. Then $f \in L^+$ and $\int f = \lim_{n\to\infty} f_n$.*

8.12. DEFINITION. A function $f : \Omega \to \bar{\mathbb{R}}$ is said to be *Lebesgue integrable* (in Ω), if there exists two functions g and h in L^+ such that $f = g - h$. In this case, the number $\int g - \int h$ denoted by $\int_\Omega f(x)\,dx$ (or simply, $\int f$) is called the *Lebesgue integral* of the function f. The linear (real) space of functions integrable in Ω is denoted by $L(\Omega)$ (or L)[4]. If $1_A \in L(\Omega)$, then the number $\mu(A) = \int 1_A$ is called the *measure* of the set $A \subset \Omega$.

[4] In this case two functions are identified, if their difference is equal to zero almost everywhere (see, in this connection, P.8.14 and P.8.18).

8.13. REMARK. One can readily check that $\int f$ depends only on f, i.e., Definition 8.12 is correct.

8.14.**P.** *Show that $\int f = 0$, if $f = 0$ almost everywhere.*

8.15.**P.** *Let $h_N(x) = \sum_{k=1}^N m_k \cdot 1_{\Pi_k}(x)$, where $m_k = \inf_{x\in\Pi_k} f(x)$ is a step function corresponding to the lower Darboux sum $\sum_{k=1}^N m_k\mu(\Pi_k)$ of a function f which is Riemann integrable. Verify that Definitions 8.8, 8.10, and 8.12 immediately imply that $f \in L^+$ (thus, $f \in L$), and, moreover, the Riemann integral equal to $\lim_{N\to\infty} \sum_{k=1}^N m_k\mu(\Pi_k)$, coincides with the Lebesgue integral $\int f(x)dx = \lim_{N\to\infty} \int h_N(x)\,dx$.*

8.16.**P.** *Let f be measurable on $[0,1]$ and bounded: $m \le f(x) \le M$. Consider the partition of the closed interval $[m, M]$ by the points $y_k : y_0 < y_1 < \cdots < y_N = M$. Let $\sigma = \max(y_k - y_{k-1})$. Consider the sum*

$$S = \sum_{k=1}^{N} y_k \mu\{x \in [0,1] \mid y_{k-1} \le f(x) \le y_k\}.$$

Prove that $\exists \lim\limits_{\sigma\to 0} S$, and this limit is the Lebesgue integral $\int f(x)\,dx$.

8.17. THEOREM (Beppo Levi, 1906; see, for instance, [**56**]). *Let $f_n \in L(\Omega)$ and $f_n(x) \uparrow f(x)$ $\forall x \in \Omega$. If there exists a constant C such that $\int f_n \le C$ $\forall n$, then $f \in L(\Omega)$ and $\lim\limits_{n\to\infty} \int f_n = \int f$.*

8.18.**P** (Compare with P.8.14)**.** *Show that $f = 0$ almost everywhere if $f \ge 0$ and $\int f = 0$.*

8.19. **P.** *Verify that if $\varphi \in L$, $\psi \in L$, then* $\max(\varphi, \psi) \in L$, $\min(\varphi, \psi) \in L$.

8.20. THEOREM (Lebesgue, 1902; see, for instance, [**56**]). *Let $f_n \in L(\Omega)$ and $f_n(x) \to f(x)$ almost everywhere in Ω. Suppose that there exists a function $g \in L(\Omega)$, which is called the majorant, such that $|f_n(x)| \le g(x)$ $\forall n \ge 1$, $\forall x \in \Omega$. Then $f \in L(\Omega)$ and $\int f = \lim\limits_{n\to\infty} \int f_n$.*

8.21. LEMMA (Fatou, 1906; see, for instance, [**56**]). *Let $f_n \in L$, $f_n \ge 0$ and $f_n \to f$ almost everywhere. If $\int f_n \le C$ $\forall n$, where $C < \infty$, then $f \in L$ and $0 \le \int f \le C$.*

8.22. THEOREM (Fischer and F. Riesz, 1907). *A space L with the norm $\|\varphi\| = \int |\varphi|$, is a Banach space*[5)].

[5)] A *norm* in a linear space X is a function $\|\cdot\| : X \ni f \longmapsto \|f\| \in \mathbb{R}$, with the following properties: $\|f\| > 0$ for $f \ne 0 \in X$, $\|0\| = 0$, $\|\lambda f\| = |\lambda| \cdot \|f\|$ for any number λ, $\|f + g\| \le \|f\| + \|g\|$. The words "the space X is endowed with the norm" mean that a notion of the *convergence* is introduced in the space X. Namely, $f_n \to f$ as $n \to \infty$, if $\rho(f_n, f) \to 0$, where $\rho(f_n, f) = \|f_n - f\|$. In this case one say that the space X is *normed.* The function introduced ρ has, as can be easily seen, the following properties: $\rho(f, g) = \rho(g, f)$, $\rho(f, h) \le \rho(f, g) + \rho(g, h)$, and $\rho(f, g) > 0$, if $f \ne g$. If a function ρ with these properties is defined on the set

$X \times X$, then this function is called the *distance* in X, and the pair (X, ρ) is called a *metric space* (in general, non-linear). It is clear that a normed space is a linear metric space. A metric space is called *complete*, if for any *fundamental sequence* $\{f_n\}_{n\geq 1}$ (this means that $\rho(f_n, f_m) \to 0$ as $n, m \to \infty$) there exists $f \in X$ such that $\rho(f_n, f) \to 0$. A complete normed space is called a *Banach space*.

PROOF. Let $\|\varphi_n - \varphi\| \to 0$ as $n, m \to \infty$. Then there exists an increasing sequence if indices $\{n_k\}_{k\geq 1}$ such that $\|\varphi_n - \varphi_m\| \leq 2^{-k}$ $\forall n > n_k$. Let us set $f_N(x) = \sum\limits_{k=1}^{N-1} |\varphi_{n_{k+1}}(x) - \varphi_{n_k}(x)|$. The sequence $\{f_N\}_{N=2}^{\infty}$ is increasing and $\int f_N \leq 1$. Using the theorem of B. Levi, we obtain that the series $\sum\limits_{k=1}^{\infty} |\varphi_{n_{k+1}}(x) - \varphi_{n_k}(x)|$ converges almost everywhere. Therefore, the series $\sum\limits_{k=1}^{\infty} |\varphi_{n_{k+1}}(x) - \varphi_{n_k}(x)|$ also converges almost everywhere. In other words, for almost all x, the limit $\lim\limits_{k\to\infty} \varphi_{n_k} = \varphi(x)$ exists[6]. Let us show that $\varphi \in L$ and $\|\varphi_n - \varphi\| \to 0$ as $n \to \infty$. We have $\forall \epsilon > 0$ $\exists N \geq 1$ such that $\int |\varphi_{n_m}(x) - \varphi_{n_k}(x)| dx \leq \epsilon$ for $n_m \geq N$, $n_k \geq N$. Using the Fatou lemma, we pass to the limit as $n_m \to \infty$. We obtain $\varphi - \varphi_{n_k} \in L$, $\int |\varphi(x) - \varphi_{n_k}(x)|\, dx \leq \epsilon$; therefore, $\varphi \in L$, and $\|\varphi - \varphi_{n_k}\| \to 0$ as $k \to \infty$. Hence, $\|\varphi - \varphi_n\| \to 0$ as $n \to \infty$, because $\|\varphi - \varphi_n\| \leq \|\varphi - \varphi_{n_k}\| + \|\varphi_{n_k} - \varphi_n\|$. □

[6] Thus, any fundamental sequence in L contains a subsequence that converges almost everywhere. We shall use this fact in Corollary 9.7 and in Lemma 10.2.

8.23.**P.** *Construct an example of a* **bounded** *sequence (compare with the example at the beginning of Section 8) which is fundamental with respect to the convergence defined by the Riemann integral but has no limit with respect to this convergence.*

HINT. For $\lambda \in [0, 1[$, consider the sequence of characteristic functions of the sets C_n that are introduced below, when constructed a Cantor set of measure $1 - \lambda$. Let $\{\lambda_n\}$ be a sequence of positive numbers such that $\sum\limits_{n=1}^{\infty} 2^{n-1}\lambda_n = \lambda \in [0, 1]$. The *Cantor set*

$C \subset [0,1]$ (see [**21, 36, 56**]) corresponding to the sequence $\{\lambda_n\}$ is constructed in the following way: $C = \cap C_n$, $C_n = [0,1] \setminus (\bigcup\limits_{m=1}^{n} I_n)$, $I_n = \bigcup\limits_{k=1}^{2^{n-1}} I_m^k$. Here, I_m^k is the kth interval of the mth rank, i.e., an interval of length λ_m whose centre coincides with the centre of the kth $(k = 1, \dots, 2^{m-1})$ closed interval of the set C_{m-1}. In other words, the Cantor set is constructed step by step in the following way. At the first step we "discard" from the closed interval $C_0 = [0,1]$ its "middle part" I_1 of length λ_1, at the second step from the two remaining closed intervals of the set $C_1 = C_0 \setminus I$ we "discard" their "middle parts" each of which has the length λ_2. At mth step $(m \geq 3)$ we "discard" from the remaining 2^{m-1} closed intervals of the set C_{m-1} their "middle parts", each one the length λ_m. It can be easily seen (verify!) that the set C is measurable and its measure $\mu(C)$ is equal to $\mu = 1 - \lambda \geq 0$. It can be rather easily shown that the set C is not countable and, moreover, there exists a one-to-one correspondence between C and $\mathbb{R}$. At first sight, this seems astonishing, since the measure $\mu(C) = 0$ for $\lambda = 1$. Nevertheless, it is true! (Prove.) The Cantor set is often used as the base of constructing some "puzzling" examples.

8.24. THEOREM (Fubini, 1907; see, for instance, [**58**]). *Let Ω_x be an open set in $\mathbb{R}^k$ and Ω_y an open set in $\mathbb{R}^m$. Suppose that $f : \Omega \ni (x,y) \longmapsto f(x,y)$ is an integrable functions in the direct product $\Omega = \Omega_x \times \Omega_y$. Then*

(1) *for almost all $y \in \Omega_x$ (respectively, $x \in \Omega_y$) the function $f(\cdot, y) : \Omega_x \ni x \longmapsto f(x,y)$ (respectively, $f(\cdot, y) : \Omega_y \ni x \longmapsto f(x,y)$) is an element of the space $L(\Omega_x)$ (respectively, $L(\Omega_y)$);*

(2) $\int\limits_{\Omega_x} f(x,\cdot)dx \in L(\Omega_y)$ *(respectively,* $\int\limits_{\Omega_y} f(\cdot,y)dy \in L(\Omega_x)$*);*

(3) $$\int\limits_{\Omega} f(x,y)dxdy = \int\limits_{\Omega_y} \left[\int\limits_{\Omega_x} f(x,y)dx \right] dy = \int\limits_{\Omega_x} \left[\int\limits_{\Omega_y} f(x,y)dy \right] dx.$$

8.25. Remark. The existence of two (iterated) integrals

$$\int\limits_{\Omega_y}\left[\int\limits_{\Omega_x} f(x,y)dx\right]dy \quad\text{and}\quad \int\limits_{\Omega_x}\left[\int\limits_{\Omega_y} f(x,y)dy\right]dx$$

implies, in general, neither their equality nor the integrability of the function f in $\Omega = \Omega_x \times \Omega_y$ (see, for instance, [**21**]). However, the following lemma holds.

8.26. Lemma. *Let f be a function defined in $\Omega = \Omega_x \times \Omega_y$. Suppose that f is measurable and $f \geq 0$. Suppose also that there exists the iterated integral $\int\limits_{\Omega_y}\left[\int\limits_{\Omega_x} f(x,y)dx\right]dy = A$. Then $f \in L(\Omega_x \times \Omega_y)$; therefore, property 3) of Theorem 8.24 holds.*

Proof. We set $f_m = \min(f, H_m)$, where $H_m = \max(h_1, \dots, h_m)$, $\{h_k\}$ is a sequence of step functions such that $h_k \to f$ almost everywhere (see Definition 8.4). Note that $f_m = \lim\limits_{k\to\infty} \min(h_k, H_m)$ almost everywhere and $|f_m| \leq |H_m|$, since $f \geq 0$. Therefore, $f_m = \lim\limits_{k\to\infty} g_{km}$ almost everywhere, where $g_{km} = \max(\min(h_k, H_m), -|H_m|)$. Furthermore, $|g_{km}| \leq |H_m| \in L$ $\forall k \geq 1$. Hence, by the Lebesgue theorem, $f \in L$. By virtue of the Fubini theorem, we have

$$\int f_m = \int\limits_{\Omega_y}\left[\int\limits_{\Omega_x} f_m(x,y)dx\right]dy \leq A.$$

Let us note that $f_n \uparrow f$. Therefore, by the B. Levi theorem, $f \in L(\Omega)$. $\square$

8.27. Theorem (see, for instance, [**56**]). *Let $f \in L(\mathbb{R})$, $g \in L(\mathbb{R})$, $F(x) = \int\limits_0^x f(t)dt$, $G(x) = \int\limits_0^x g(t)dt$. Then*

$$\int\limits_a^b F(x)g(x)dx + \int\limits_a^b f(x)G(x)dx = F(b)G(b) - F(a)G(a).$$

In this case the function $F(x)$ has for almost all $x \in \mathbb{R}$ the derivative $F'(x) = \lim_{\sigma \to 0} (F(x+\sigma) - F(x))/\sigma$, and $F'(x) = f(x)$ almost everywhere.

9. The spaces L^p and L^p_{loc}

9.1. Definition (F. Riesz). Let $1 \le p \le \infty$. By the space $L^p(\Omega)$ (or simply L^p) of *functions integrable in pth power* we call the complex space of measurable functions[1)] f defined in Ω and such that $|f|^p \in L(\Omega)$. If $f \in L^1(\Omega)$, then the integral of f is defined by the formula

$$\int f = \int \Re f + i \int \Im f.$$

[1)] More exactly, of the classes of functions $\{f\} : \Omega \to \mathbb{C}$, where $g \in \{f\} \Longleftrightarrow g = f$ almost everywhere.

9.2. Lemma. *Let $p \in [1, \infty[$. Then the mapping*

$$\| \cdot \|_p : L^p \ni f \longmapsto \|f\|_p = \left(\int_\Omega \|f(x)\|^p dx \right)^{1/p}, \tag{9.1}$$

which will sometimes be noted by $\| \cdot \|_{L^p}$ is a norm[2)].

[2)] That is the properties of a norm (see note 5 in Section 8) hold.

Proof. It is not obvious only the validity of the triangle inequality, i.e., the inequality

$$\|f + g\|_p \le \|f\|_p + \|g\|_p \tag{9.2}$$

which (in case of norm (9.1)) is called the *Minkowski inequality*. It is trivial for $p = 1$. Let us prove it for $p > 1$, using the known [**36**] *Hölder inequality*

$$\|f \cdot g\|_1 \le \|f\|_p \cdot \|g\|_q, \quad \text{where} \quad 1/p + 1/q = 1, \; p > 1. \tag{9.3}$$

We have

$$\begin{aligned} \int |f+g|^p &\le \int |f+g|^{p-1}|f| + \int (|f+g|^{p-1}|g|) \\ &\le \left[\int |f+g|^{(p-1)\cdot q} \right]^{1/q} \left\{ \left[\int |f|^p \right]^{1/p} + \left[\int |g|^p \right]^{1/p} \right\}. \end{aligned}$$

However,

$$\left[\int |f+g|^{(p-1)\cdot q}\right]^{1/q} = \left[\int |f+g|^p\right]^{1-(1/p)}.$$

□

Similarly to the proof of Theorem 8.22, one can prove

9.3. LEMMA. *Let $1 \le p < \infty$. The space L^p with norm* (9.1) *is a Banach space.*

9.4. LEMMA. *The complexification of the space of step functions*[3)] *is dense in L^p, $1 \le p < \infty$.*

[3)] The *complexification* of a real linear space X is the complex linear space of elements of the form $f = g + ih$, where g and h are elements of X.

PROOF. It is sufficient to prove that $\forall f \in L^p$, where $f \ge 0$, there exists a sequence $\{h^k\}$ of step functions such that

$$\|f - h_k\|_p \to 0 \text{ as } k \to \infty. \tag{9.4}$$

In case $p = 1$, we take the sequence $\{h_k\}_{k=1}^\infty$ such that $h_k \uparrow f \in L^+$ and $\int h_k \to \int f$. Then we obtain (9.4). If $1 < p < \infty$, then we set $E_n = \{x \in \Omega \mid 1/n \le f(x) \le n\}$, where $n \ge 1$ and $f_n(x) = 1_{E_n}(x) \cdot f(x)$ (1_{E_n} is the characteristic function of E_n). We have $f_n \uparrow f$; hence, $(f - f_n)^p \downarrow 0$. By the B. Levi theorem, $\|f - f_n\|_p = \left(\int_\Omega |f(x) - f_n(x)|^p dx\right)^{1/2} \to 0$ for $n \to \infty$. Therefore, $\forall \epsilon > 0\ \exists n \ge 1$ such that $\|f - f_n\|_p < \epsilon/2$. Let us fix this n. Note that $\int 1_{E_n} = \int 1_{E_n}^p \le \int n^p |f|^p < \infty$. By virtue of the Hölder inequality, $\int f_n = \int 1_{E_n} f \le \left(\int 1_{E_n}^q\right)^{1/q} \int (f^p)^{1/p} < \infty$. Since $f_n \in L(\Omega)$ and $f_n(x) \in [0, n]\ \forall x \in \Omega$, there exists a sequence $\{h_k\}$ of step functions defined in Ω with values in $[0, n]$ such that $\lim_{k\to\infty} \int |f_n - h_k| = 0$. Therefore,

$$\|f_n - h_k\|_p = \left[\int |f_n - h_k|^p\right]^{1/p} = \left[\int \left(|f_n - h_k|^{p-1} |f_n - h_k|\right)\right]^{1/p}$$
$$\le n^{1-(1/p)} \left[\int |f_n - h_k|\right]^{1/p} \to 0 \text{ for } k \to \infty.$$

Choose K such that $\|f_n - h_k\|_p < \epsilon/2$ for $k \geq K$. Then

$$\|f - h_k\|_p \leq \|f - f_n\|_p + \|f_n - h_k\| < \epsilon \quad \forall k \geq K.$$

$\square$

9.5. THEOREM. *Let $f \in L^1(\Omega)$ and $f = 0$ almost everywhere outside some $K \Subset \Omega$. Let $\rho > 0$ be the distance between K and $\partial\Omega$. Then for any $\epsilon \in]0, \rho]$, the function*

$$R_\epsilon(f) = f_\epsilon : \Omega \ni x \longmapsto f_\epsilon(x) = \int f(y)\delta_\epsilon(x - y)dy, \tag{9.5}$$

where δ_ϵ is defined in (3.2), *belongs to the space $C_0^\infty(\Omega)$. Moreover*[4],

$$\lim_{\epsilon \to 0} \|f - f_\epsilon\|_p = 0, \quad 1 \leq p < \infty. \tag{9.6}$$

[4] Function (9.5) is called the (Steklov) *smoothing function* of the function f.

PROOF. Obviously, $f_\epsilon \in C_0^\infty(\Omega)$. Let us prove (9.6). By virtue of Lemma 9.4, $\forall \eta > 0$ there exists a function $h = h_1 + ih_2$, where h_1 and h_2 are step functions, such that $\|f - h\|_p < \eta$. We have: $\|f - f_\epsilon\|_p \leq \|f - h\|_p + \|h - R_\epsilon(h)\|_p + \|R_\epsilon(f - h)\|_p$. Let s show that $\|R_\epsilon(g)\|_p \leq \|g\|_p$. For $p = 1$ this is obvious:

$$\int\limits_\Omega \left[\int\limits_\Omega |g(y)| \cdot \delta_\epsilon(x - y)dy \right] dx$$
$$= \int\limits_\Omega \left[\int\limits_\Omega \delta_\epsilon(x - y)dx \right] |g(y)|dy = \int\limits_\Omega |g(y)|dy.$$

If $p > 1$, then by inequality (9.3):

$$\begin{aligned}
\|R_\epsilon(g)\|_p^p &= \int_\Omega |g_\epsilon(x)|^p dx \\
&\le \int_\Omega \left[\int_\Omega (\delta_\epsilon(x-y))^{(p-1)/p)} (\delta_\epsilon(x-y)^{1/p} |g(y)|) dy \right]^p dx \\
&\le \int_\Omega \left[\left(\int_\Omega \delta_\epsilon(x-y) dy \right)^{(p-1)/p} \left(\int_\Omega \delta_\epsilon(x-y) |g(y)|^p dy \right)^{1/p} \right]^p dx \\
&= \int_\Omega \left[\int_\Omega \delta_\epsilon(x-y) |g(y)|^p dy \right] dx \\
&= \int_\Omega \left[\int_\Omega \delta_\epsilon(x-y) dx \right] |g(y)|^p dy = \int_\Omega |g(y)|^p dy.
\end{aligned}$$

Thus, $\|f - f_\epsilon\|_p \le 2\eta + \|h - R_\epsilon(h)\|_p$. By virtue of (8.1), $h = \sum\limits_{k=1}^N c_k \cdot 1_{\Pi_k}$, where $c_k \in \mathbb{C}$; hence,

$$\begin{aligned}
\|h - R_\epsilon(h)\|_p^p &= \left| \int_\Omega | \sum_{k=1}^N c_k \cdot (1_{\Pi_k} - R_\epsilon(1_{\Pi_k})) \right|^p dx \\
&\le \left(\sum_{k=1}^N |c_k| \right)^p \max_k \int_\Omega |1_{\Pi_k} - R_\epsilon(1_{\Pi_k}) dx \le C \cdot \epsilon,
\end{aligned}$$

because $(1_{\Pi_k} - R_\epsilon(1_{\Pi_k})) = 0$ outside the ϵ-neighbourhood of the parallelepiped Π_k. Taking $\epsilon < \eta^p/C$, we obtain $\|h - R_\epsilon(h)\|_p < \eta$. $\square$

9.6. Corollary. *$C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$, $1 \le p \le \infty$.*

Proof. Let $g \in L^p(\Omega)$. Note that $\forall \eta > 0$ $\exists K \Subset \Omega$ such that $\|g - g \cdot 1_K\|_p < \eta$. By Theorem 9.5 there exists $\epsilon > 0$ such that $\|g \cdot 1_K - R_\epsilon(g \cdot 1_K)\|_p < \eta$. $\square$

9.7. Corollary. *Let $f \in L^1(\Omega)$, $f = 0$ almost everywhere outside $K \Subset \Omega$. Then there exists a sequence of functions $f_m \in$*

$C_0^\infty(\Omega)$ such that $f_m \to f$ almost everywhere as $m \to \infty$, if $|f| \le M$ almost everywhere.

PROOF. By virtue of (9.5)–(9.6), $\|f - R_\epsilon(f)\|_1 \to 0$ as $\epsilon \to 0$. Therefore, according to note 6 in Section 8, there exists a subsequence $\{f_m\}$ of the sequence $\{R_\epsilon(f)\}_{\epsilon\to 0}$ such that $f_m \to f$ almost everywhere. The estimate $|f_m| \le M$ is obvious. □

9.8.**P.** [5] *Prove that $\|u - R_\epsilon(u)\|_C \to 0$ as $\epsilon \to 0$, if $u \in C_0(\Omega)$.*

[5] Here and below, $\|f\|_C = \sup_{x\in\Omega} |f(x)|$ for $f \in C(\bar{\Omega})$.

9.9. DEFINITION. $L^\infty(\Omega)$ is the *space of essentially bounded functions* in Ω, i.e., the space of measurable functions $f : \Omega \to \mathbb{C}$ such that

$$\|f\|_\infty = \inf_{\omega\in\Omega} \sup_{x\in\omega} |f(x)| < \infty, \quad \mu(\Omega \setminus \omega) = 0. \tag{9.7}$$

Condition (9.7) means that the function f is bounded almost everywhere, i.e., $\exists M < \infty$ such that $|f(x)| \le M$ almost everywhere and $\|f\|_\infty = \inf M$.

One can readily prove

9.10. LEMMA. *The space $L^\infty(\Omega)$ with the norm* (9.7) *is a Banach space,*

9.11. REMARK. The symbol ∞ in the designation of the space and the norm (9.7) is justified by the fact that $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$, if $\Omega \Subset \mathbb{R}^n$. This fact is proved, for instance, in [**71**].

9.12. DEFINITION. Let X be a normed space with a norm $\|\cdot\|$. Then X' denotes the space of continuous linear functional on X. The space X' is called *dual* to X.

One can readily prove

9.13. PROPOSITION. *The space X' equipped with the norm*

$$\|f\|' = \sup_{x\in X} \frac{|\langle f, x\rangle|}{\|x\|}, \quad f \in X',$$

is a Banach space. Here, $\langle f, x\rangle$ is that value of f at $x \in X$.

9.14. THEOREM (F. Riesz, 1910). *Let $1 \le p < \infty$. Then $(L^p)' = L^q$, where $1/p + 1/q = 1$ ($q = \infty$ for $p = 1$). More exactly:*

1) *$\forall f \in L^q(\Omega)$ $\exists F \in (L^p(\Omega))'$, i.e., a linear continuous functional F on $L^p(\Omega)$ such that*

$$\langle F, \varphi \rangle = \int\limits_\Omega f(x)\varphi(x)dx \quad \forall \varphi \in L^p(\Omega); \tag{9.8}$$

2) *$\forall F \in (L^p(\Omega))'$ there exists a unique*[6] *element (function) $f \in L^q(\Omega)$ such that (9.8) holds;*

[6] *See note 4 in Section 8.*

3) *the correspondence $I : (L^p)' \ni F \longmapsto f \in L^q$ is an isometric isomorphism of Banach spaces, i.e., the mapping I is linear bijective and $\|IF\|_q = \|F\|'_p$.*

PROOF. Assertion 1) as well as the estimate $\|F\|'_p \le \|f\|_q$ are obvious for $p = 1$. For $p > 1$ one should use the Hölder inequality. Assertion 2) as well as the estimate $\|F\|'_p \ge \|f\|_q$ are proven, for instance, in [**71**]. Assertion 3) follows from 1) and 2). □

9.15. DEFINITION. Let $p \in [1, \infty]$. Then $L^p_{loc}(\Omega)$ (or simply L^p_{loc}) denotes the *space of functions locally integrable in* pth *power* $f : \Omega \to \mathbb{C}$, i.e., of the functions such that $f \cdot 1_K \in L^p(\Omega)$ $\forall K \Subset \Omega$. We introduce in $L^p_{loc}(\Omega)$ the *convergence*: $f_j \to f$ in $L^p_{loc}(\Omega)$ if and only if $\|1_K \cdot (f_j - f)\|_p \to 0$ as $j \to \infty$ $\forall K \Subset \Omega$.

Let us note the obvious fact: if $1 < r < s < \infty$, then

$$PC \subsetneq L^\infty_{loc} \subsetneq L^s_{loc} \subsetneq L^r_{loc} \subsetneq L^1_{loc}.$$

10. Functions of L^1_{loc} as linear functional on C_0^∞

The idea of representability (i.e., of determination) of a function by its "averagings", outlined in Sections 1–2 can now be written in a rather general form as the following

10.1. THEOREM. *Any function $f \in L^1_{loc}(\Omega)$ can be uniquely*[1] *reconstructed by the linear functional*

$$\langle f, \cdot \rangle : C_0^\infty(\Omega) \ni \varphi \longmapsto \langle f, \varphi \rangle = \int\limits_\Omega f(x)\varphi(x)dx \in \mathbb{C} \tag{10.1}$$

(i.e., by the set of the numbers $\left\{\langle f,\varphi\rangle\big|_{\varphi\in C_0^\infty(\Omega)}\right\}$*). Moreover, the correspondence* $f \longleftrightarrow \langle f,\cdot\rangle$, $f\in L^1_{loc}$*, is an isomorphism.*

[1] As an element of the space $L^1_{loc}(\Omega)$ (see note 4 in Section 8).

PROOF. Suppose that two functions f_1 and f_2 correspond to one functional. Then $\int(f_1-f_2)\varphi=0\ \forall\varphi\in C_0^\infty$. This, by virtue of Lemma 10.2 below, implies that $f_1=f_2$ almost everywhere. □

10.2. LEMMA. *Let* $f\in L^1_{loc}(\Omega)$*. If* $\int_\Omega f(x)\varphi(x)dx=0\ \forall\varphi\in C_0^\infty(\Omega)$*, then* $f=0$ *almost everywhere.*

PROOF. Let $\omega\Subset\Omega$. Note that $|f|\cdot 1_\omega=f\cdot g$, where $g(x)=1_\omega\cdot\exp[-i\arg f(x)]$. (If f is a real function, then $g(x)=-\operatorname{sgn} f(x)\cdot 1_\omega(x)$.) According to Corollary 9.7, there exists a sequence of functions $\varphi_n\in C_0^\infty(\Omega)$, such that almost everywhere in Ω $f\cdot\varphi_n\to f\cdot g$ as $n\to\infty$, and $|\varphi_n|\le 1$. Since $\int_\omega|f|=\int_\Omega f\cdot g$ and by the Lebesgue theorem $\int_\Omega f\cdot g=\lim\limits_{n\to\infty}\int_\Omega f\cdot\varphi_n$, we have $\int_\omega|f|=0$, because $\int_\Omega f\cdot\varphi_n=0$. Thus, $f=0$ almost everywhere in ω. Hence, by virtue of arbitrariness of $\omega\Subset\Omega$, $f=0$ almost everywhere in Ω. □

11. Simplest hyperbolic equations. Generalized Sobolev solutions

In this section we illustrate one of the main achievements of the theory of distributions on the example of the simplest partial differential equation $u_t+u_x=0$,[1] which is sometimes called the transfer equation. It concerns a new meaning of solutions of differential equations, more exactly, a new (extended) meaning of differential equations. This meaning allows us to consider correctly some important problems of mathematical physics which have no solutions in the usual sense. This new approach to the equations of mathematical physics and their solutions, designed by S.L. Sobolev in 1935 (see, for instance, [**62**]) under the title "generalized solutions", allows, in particular, to prove the existence and uniqueness theorem for the generalized solution of the Cauchy problem:

$$Lu\equiv u_t+u_x=0,\quad (x,t)\in\mathbb{R}^2_+=\{(x,t)\in\mathbb{R}^2\mid t>0\},\qquad(11.1)$$

$$u\big|_{t=0}=f(x),\quad x\in\mathbb{R}\qquad(11.2)$$

for equation (11.1) for any function $f \in PC(\mathbb{R})$ (and even $f \in L^1_{loc}$; see Theorem 11.10 below). The theorem is also valid (see P.11.11) on the continuous dependence of the solution of this problem on $f \in L^1_{loc}(\mathbb{R})$.

[1] Here, u_t and u_x denote the partial derivatives of the function $u(x,t)$ with respect to t and x.

Let us clarify the essence of the problem. Equation (11.1) is equivalent to the system

$$u_t + u_x \cdot dx/dt = 0, \quad dx/dt = 1.$$

Therefore, along the line $x = t + a$, where a is a real parameter, we have $du(t+a,t)/dt = 0$. In other words, $u(t+a,t) = u(a,0)\ \forall t$. Thus, the function $f(x) = \lim\limits_{t\to+0} u(x,t)$ must necessary be continuous; in this case $u(x,t) = f(x-t)$.[2] If f is differentiable, then $u(x,t) = f(x-y)$ is a solution of problem (11.1)–(11.2). However, this problem has no solution (differentiable or even continuous), if f is discontinuous, for instance, if $f(x) = \theta(x)$, where $\theta : \mathbb{R} \ni x \longmapsto \theta(x) \in \mathbb{R}$ is the *Heaviside function*, i.e.,

$$\theta(x) = 1 \ \text{ for } \ x \geq 0 \quad \text{and} \quad \theta(x) = 0 \ \text{ for } \ x < 0. \tag{11.3}$$

[2] This formula implies that the graph of the function $x \longmapsto u(x,t)$ for any fixed t can be obtained by the transition (shift) of the graph of the function f to the right along the axis x on the distance t. It is the reason why equation (11.1) is sometimes called the transfer equation.

However, consideration of problem (11.1)–(11.2) with the initial function (11.3) is justified at least by the fact that this problem arises (as minimum, on the formal level) when studying the propagation of the plane sonic waves in a certain medium. The appropriate process is described by the so-called *acoustic system of differential equations*

$$u_t + \frac{1}{\rho} p_x = 0,\ p_t + \rho \cdot c^2 u_x = 0, \quad \rho > 0,\ c > 0. \tag{11.4}$$

Here, ρ is the density, c is the characteristic of the compressible medium, and $u = u(x,t)$ and $p = p(x,t)$ are the velocity and the pressure at the instant t at the point x. Setting

$$\alpha = u + p/(\rho \cdot c), \quad \beta = u - p/(\rho \cdot c),$$

we obtain the equivalent system $\alpha_t + c\alpha_x = 0$, $\beta_t - c\beta_x = 0$ of two transfer equations. Thus, problem (11.1)–(11.2) with the initial function (11.3) arises when considering the propagation of sonic waves, say, for the initial velocity $u(x,t) = \theta(x)$ and zero initial pressure.

11.1.**P.** *Show that any solution of the class C^1 of system* (11.4) *can be represented in the form*

$$\begin{aligned} &u(x,t) = [\varphi(x-ct) + \psi(x+ct)]/2, \\ &p(x,t) = [\varphi(x-ct) - \psi(x+ct)]/2, \quad \textit{where} \ \ \varphi \in C^1, \ \psi \in C^1. \end{aligned} \tag{11.5}$$

11.2.**P.** *Show that the following theorem is valid.*

11.3. THEOREM. $\forall f \in C^1(\mathbb{R})\ \forall F \in C(\bar{\mathbb{R}}^2_+)$ *the Cauchy problem*

$$u_t + u_x = F(x,t) \quad \textit{in} \ \ \mathbb{R}^2_+, \qquad u\big|_{t=0} = f(x), \qquad x \in \mathbb{R}$$

has a unique solution $u \in C^1(\bar{\mathbb{R}}^2_+)$.

As has been said, for $f(x) = \theta(x)$ problem (11.1)–(11.2) has no regular solution (i.e., a solution in the usual sense of this word); nevertheless the arguments which lead to the formula $u(x,t) = f(x-t)$ as well as this formula suggest to call by the solution of problem (11.1)–(11.2) the function $f(x-t)$ for whatever function $f \in PC(\mathbb{R})$ (and even $f \in L^1_{loc}(\mathbb{R})$), especially, as the following lemma holds.

11.4. LEMMA. *Let $f \in L^1_{loc}(\mathbb{R})$ and $\{f_n\}$ be a sequence of functions $f_n \in C^1(\mathbb{R})$ such that*[3)]

$$f_n \to f \quad \textit{in} \ \ L^1_{loc}(\mathbb{R}) \qquad \textit{as} \ \ n \to \infty.$$

Then the function $u : \mathbb{R}^2_+ \ni (x,t) \longmapsto u(x,t) = f(x-t)$ belongs to $L^1_{loc}(\mathbb{R}^2_+)$, and $u = \lim\limits_{n\to\infty} u_n$ in $L^1_{loc}(\mathbb{R}^2_+)$, where $u(x,t) = f(x-t)$.[4)]

[3)] According to Lemma 11.5 below, such a sequence exists.

[4)] Note that $u_n(x,t)$ is a solution of problem (11.1)–(11.2), if $u_n\big|_{t=0} = f_n(x)$.

PROOF. Since $f = f^1 + if^2$ and $f^k = f^k_+ - f^k_-$, where $f^k_\pm = \max(\pm f^k, 0)$, it is sufficient to consider the case $f \geq 0$. Let us make the change of the variables $(x,t) \longmapsto (y,t)$, where $y = x - t$. Note that $u(x,t) = f(y)$ and

$$\int_a^b \left(\int_c^d u(x,t)dx \right) dt \le \int_a^b \left(\int_{c-d}^{d-a} f(y)dy \right) dt < \infty$$

for any a, b, c, d such that $0 < a < b$, $c < d$. By virtue of Lemma 8.26, this implies $u \in L^1_{loc}(\mathbb{R}^2_+)$. Furthermore, for the same a, b, c, and d

$$\int_a^b \left(\int_c^d |u_n(x,t) - u(x,t)|dx \right) dt \le (b-a) \int_{c-b}^{d-a} |f_n(y) - f(y)|dy \to 0$$

as $n \longrightarrow \infty$. $\square$

11.5. Lemma. *$\forall f \in L^1_{loc}(\mathbb{R})$ there exists a sequence of functions $f_n \in C^\infty(\mathbb{R})$ converging to f in $L^1_{loc}(\mathbb{R})$.*

Proof. Let $\{\varphi_\mu\}_{\mu=1}^\infty$ be the partition of unity in $\mathbb{R}$ (see Section 3), $\psi_\mu \in C_0^\infty(\mathbb{R})$ and $\psi_\mu \cdot \varphi_\mu = \varphi_\mu$. We have $\psi_\mu \cdot f \in L^1(\mathbb{R})$. By Theorem 9.5, there exists a sequence $\{f_n^\mu\}_{\mu=1}^\infty$ of functions $f_n^\mu \in C^\infty(\mathbb{R})$ such that for any fixed μ: $\lim\limits_{n\to\infty} \|\psi_\mu f - f_n^\mu\|_1 = 0$. Setting $f_n(x) = \sum\limits_{\mu=1}^\infty \varphi_\mu(x) f_n^\mu(x)$, $x \in \mathbb{R}$, we have $f_n \in C^\infty(\mathbb{R})$. Note that $\forall c > 0\ \exists M \in \mathbb{N}$ such that $\sum\limits_{\mu=1}^\infty \varphi_\mu(x) = \sum\limits_{\mu=1}^M \varphi_\mu(x)$ for $|x| < c$. Hence,

$$\int_{-c}^c |f(x) - f_n(x)|dx = \int_{-c}^c \left| f(x) - \sum_{\mu=1}^M \varphi_\mu(x) f_n^\mu(x) \right| dx$$

$$+ \int_{-c}^c \left| \sum_{\mu=1}^M \varphi_\mu(\psi_\mu f - f_n^\mu) \right| dx \le \sum_{\mu=1}^M \int_{-c}^c |\psi_\mu f - f_n^\mu| dx.$$

Therefore,

$$\lim_{n\to\infty} \int_{-c}^c |f - f_n|dx \le \sum_{\mu=1}^M \lim_{n\to\infty} \int_{-c}^c |\psi_\mu f - f_n^\mu|dx = 0,$$

because $\lim\limits_{n\to\infty} \|\psi_\mu f - f_n^\mu\|_1 = 0$. $\square$

The definition of the solution of problem (11.1)–(11.2), where $f \in L^1_{loc}$, with the help of the formula $u(x,t) = f(x-t)$ is rather tempting, however, let us note that it has a serious defect: with the help of a concrete formula, one can define the solution of only a small class of problems. Lemma 11.4 suggests a definition free of this defect.

11.6. Definition. Let $f \in L^1_{loc}(\mathbb{R})$. We say that $u \in L^1_{loc}(\mathbb{R}^2_+)$ is a generalized solution of problem (11.1)–(11.2), if there exists a sequence of solutions $u_n \in C^1(\bar{\mathbb{R}}^2_+)$ of equation (11.1) such that, as $n \to \infty$,

$$u_n \to u \ \text{ in } \ L^1_{loc}(\mathbb{R}^2_+) \qquad \text{and} \qquad u_n\big|_{t\to 0} \to f \ \text{ in } \ L^1_{loc}(\mathbb{R}).$$

Approximative approach to the definition of a generalized solution can be applied to a large class of problems. So, it has been above constructed (but was not named), for instance, the generalized solution of the equation $\Delta E(x) = \delta(x)$ (see (7.9)) as well as the generalized solution of the problem $\Delta P = 0$ in $\mathbb{R}^2_+$, $P(x,0) = \delta(x)$ (see Remark 5.4). However, the approximative definition, in spite of technical convenience, also has an essential shortage: it does not show the real mathematical object, the "generalized" differential equation, whose immediate solution is the defined "generalized solution".

It is reasonable to search for the appropriate definition of the generalized solutions of differential equations (and appropriate "generalized" differential equations), by analyzing the deduction of the equations of mathematical physics (in the framework of one or another conception of the continuous medium). The analysis fulfilled in Sections 1–2, Lemma 10.2, and the Ostrogradsky–Gauss formula (7.2) suggest the suitable definition (as will be seen from Proposition 11.8).

11.7. Definition. Let $f \in L^1_{loc}(\mathbb{R})$. A function $u \in L^1_{loc}(\mathbb{R}^2_+)$ is called a generalized solution of problem (11.1)–(11.2), if it satisfies the following equation (so-called integral identity (in φ))

$$\int\limits_{\mathbb{R}^2_+} (\varphi_t + \varphi_x) u(x,t)\,dx\,dt + \int\limits_{\mathbb{R}} \varphi(x,0) f(x)\,dx = 0 \quad \forall \varphi \in C^1_0(\bar{\mathbb{R}}^2_+). \tag{11.6}$$

11.8. Proposition. *If* $u \in C^1(\bar{\mathbb{R}}^2_+)$, *then* (11.6) *is equivalent to* (11.1)–(11.2).

PROOF. Let $\varphi \in C_0^1(\bar{\mathbb{R}}_+^2)$ and let Ω be a bounded domain in $\mathbb{R}_+^2$ with the boundary $\Gamma = \partial\Omega$. Formula (7.2) implies that

$$\int\limits_{\Omega} (u_t + u_x)\varphi \, dx \, dt + \int\limits_{\Omega} (\varphi_t + \varphi_x) u \, dx \, dt = \int\limits_{\partial\Omega} (\varphi \cdot u)[\cos(\nu, t) + \cos(\nu, x)] \, d\Gamma. \quad (11.7)$$

If $\operatorname{supp}\varphi \subset \bar{\Omega}$ and (see Fig.) $(\operatorname{supp}\varphi \cap \partial\Omega) \subset \mathbb{R}_x = \{(x,t) \in \mathbb{R}^2 \mid t = 0\}$, then formula (11.7) can be rewritten in the form

$$\int\limits_{\mathbb{R}_+^2} (u_t + u_x)\varphi dx dt + \int\limits_{\mathbb{R}_+^2} (\varphi_t + \varphi_x) u dx dt = -\int\limits_{\mathbb{R}} (\varphi u)\big|_{t=0} dx. \quad (11.8)$$

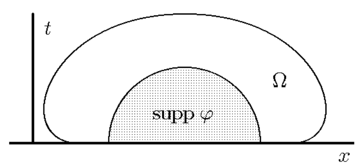
Furthermore, by virtue of Lemma 10.2,

$$(11.1) \iff \int\limits_{\mathbb{R}_+^2} (u_t + u_x)\varphi \, dx \, dt = 0 \quad \forall \varphi \in C_0^1(\bar{\mathbb{R}}_+^2)$$

and

$$(11.2) \iff \int\limits_{\mathbb{R}} f(x)\varphi(x,0) dx = \int\limits_{\mathbb{R}} u\big|_{t=0} \cdot \varphi(x,0) dx \ \forall \varphi \in C_0^1(\bar{\mathbb{R}}_+^2).$$

This and (11.8) imply that (11.6)$\iff$(11.1)–(11.2). □



Proposition 11.8 shows that Definition 11.7 is consistent with the definition of an ordinary (differentiable or, as one says, regular) solution of problem (11.1)–(11.2). The following Theorem 11.10 justifies the new features appearing in Definition 11.7 and shows that the integral equality (11.6) is the same "generalized" differential equation which has been spoken about.

11.9. REMARK. The proof of Proposition 11.8 comes from the deduction of the Euler–Lagrange equation and the transversality conditions in calculus of variations proposed by Lagrange (see, for instance, [**56**]).

11.10. THEOREM. *$\forall f \in L^1_{loc}(\mathbb{R})$ problem* (11.1)–(11.2) *has a (unique) generalized solution $u \in L^1_{loc}(\mathbb{R}_+^2)$.*

PROOF. First, we prove the existence. Since the function

$$u : \mathbb{R}^2_+ \ni (x,t) \longmapsto u_n(x,t) = f(x-t)$$

is a regular solution of equation (11.1) and satisfies the initial condition $u_n\big|_{t=0} = f_n(x)$, by virtue of proposition 11.8 we have

$$\int\limits_{\mathbb{R}^2_+} (\varphi_t + \varphi_x) \cdot u_n \, dx \, dt + \int\limits_{\mathbb{R}} f_n(x)\varphi(x,0)\, dx = 0 \qquad \forall \varphi \in C^1_0(\bar{\mathbb{R}}^2_+). \tag{11.9}$$

On the other hand, by Lemma 11.4, the sequence $\{u_n\}_{n=1}^{\infty}$ tends in $L^1_{loc}(\mathbb{R}^2_+)$ to the function $u \in L^1_{loc}(\mathbb{R}^2_+)$ such that $u(x,t) = f(x-t)$. It remains to verify that the function u satisfies (11.6). For this purpose we note: $\forall \varphi \in C^1_0(\bar{\mathbb{R}}^2_+)$ $\exists a_\varphi > 0$ and $b_\varphi > 0$ such that

$$\operatorname{supp} \varphi \subset \{(x,t) \in \mathbb{R}^2 \mid |x| \le a_\varphi, 0 \le t \le b_\varphi\}.$$

Therefore,

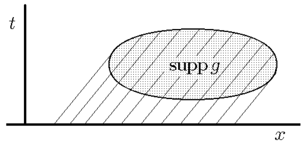
$$\begin{aligned} &\left| \int\limits_{\mathbb{R}^2_+} (u_n(x,t) - u(x,t))(\varphi_t + \varphi_x) dx dt \right| \\ &\qquad \le [\max_{(x,t)} |\varphi_t + \varphi_x|] \cdot \int\limits_0^{b_\varphi} \left(\int\limits_{-a_\varphi}^{a_\varphi} |f_n(x-t) - f(x-t)| dx \right) dt \\ &\qquad\quad \le M_\varphi \cdot b_\varphi \int\limits_{|x| \le a_\varphi + b_\varphi} |f_n(x) - f(x)| dx \to 0 \ \text{ for } \ n \to \infty. \end{aligned}$$

Taking into account (11.9), we obtain (11.6).

Now prove the uniqueness. Let u_1 and u_2 be two generalized solutions of problem (11.1)–(11.2). Then their difference $u = u_1 - u_2$ satisfies the relation $\int_{\mathbb{R}^2_+} (\varphi_t + \varphi_x) u dx dt = 0$ $\forall \varphi \in C^1_0(\bar{\mathbb{R}}^2_+)$. Show that $u(x,t) = 0$ almost everywhere. By virtue of Lemma 10.2, it is sufficient to show that the equation

$$\varphi_t + \varphi_x = g(x,t), \quad (x,t) \in \mathbb{R}^2_+ \tag{11.10}$$

has a solution $\varphi \in C^1_0(\bar{\mathbb{R}}^2_+)$ for any $g \in C^\infty_0(\mathbb{R}^2_+)$. However, this follows from P.11.2. Indeed, let $T > 0$ be such that $g(x,t) \equiv 0$, if $t \ge T$. We set



$$\varphi(x,t) = \int_T^t g(x-t+\tau,\tau)\,d\tau.$$

Obviously, (see Fig.) $\varphi \in C_0^1(\bar{\mathbb{R}}_+^2)$ and φ is a solution of (11.10). $\square$

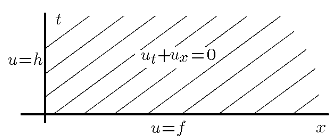
11.11.**P.** *Prove that the generalized solution of problem* (11.1)–(11.2) *depends continuously in* $L^1_{loc}(\mathbb{R}^2_+)$ *on the initial function* $f \in L^1_{loc}(\mathbb{R})$.

11.12.**P.** *Analyzing the proof of Theorem 11.10, prove that Definition 11.7 is equivalent to Definition 11.6*

11.13.**P.** *Verify directly that the function* $u(x,t) = \theta(x-t)$ *is a solution of problem* (11.1)–(11.2) *in the sense of Definition 11.7, if* $f(x) = \theta(x)$.

In the exercises below, we assume $Q = \{(x,t) \in \mathbb{R}^2 \mid x > 0, t > 0\}$.

11.14.**P.** *Consider the problem*



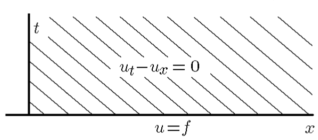
$$u_t + u_x = 0 \quad in \quad Q, \tag{11.11}$$

$$u\big|_{t=0} = f(x), \quad x > 0, \tag{11.12}$$

$$u\big|_{x=0} = h(t), \quad t > 0. \tag{11.13}$$

This problem is called mixed, because it simultaneously includes the initial condition (11.12) *and the boundary condition* (11.13). *Show that problem* (11.11)–(11.13) *has a (unique) solution* $u \in C^1(\bar{Q})$ *if and only if* $f \in C^1(\bar{\mathbb{R}}_+)$, $h \in C^1(\bar{\mathbb{R}}_+)$, *ans* $f(0) = h(0)$, $f'(0) = -h'(0)$.

11.15.**P.** *Show that the following problem*



$$u_t - u_x = 0 \quad in \quad Q, \tag{11.14}$$

$$u\big|_{t=0} = f(x), \quad x > 0 \tag{11.15}$$

has a (unique) solution $u \in C^1(\bar{Q})$ *if and only if* $f \in C^1(\bar{\mathbb{R}}_+)$. *Compare with problem* (11.11)–(11.13). *Compare the characteristics, i.e., the families of the straight lines* $dx/dt = 1$ *and* $dx/dt = -1$ *(shown in figures), along which the solutions of equations* (11.11) *and* (11.14) *are constant.*

11.16. **P.** *Consider a mixed problem for the system of acoustic equations*

$$u_t + (1/\rho)p_x = 0, \quad p_t + \rho c^2 u_x = 0, \quad (x,t) \in Q, \tag{11.16}$$

$$u\big|_{t=0} = f(x), \quad p\big|_{t=0} = g(x), \quad x > 0, \tag{11.17}$$

$$p\big|_{x=0} = h(t), \quad t > 0, \tag{11.18}$$

where f, g, *and* h *are functions from* $C^1(\bar{\mathbb{R}}_+)$.

(1) *Draw the level lines of the functions* $u \pm (1/\rho c)p$.
(2) *Show that problem* (11.16)–(11.18) *has a (unique) solution* $u \in C^1(\bar{Q})$, $p \in C^1(\bar{Q})$ *if and only if*

$$h(0) = g(0) \quad \text{and} \quad f'(0) + (1/\rho c^2)h'(0) = 0. \tag{11.19}$$

Show also that this solution (u, p) *can be represented by formulae* (11.5), *where*

$$\varphi(y) = f(y) + (1/\rho c)g(y), \quad \psi(y) = f(y) - (1/\rho c)g(y), \quad \text{if} \ \ y > 0 \tag{11.20}$$

and

$$\varphi(y) = (2/\rho c)h(-y/c) + f(-y) - (1/\rho c)g(-y), \quad \text{if} \ \ y \leq 0. \tag{11.21}$$

11.17. Remark. Often instead of system (11.4) of the acoustic equations, the following second order equation is considered

$$\partial^2 p/\partial t^2 - c^2 \cdot \partial^2 p/\partial x^2 = 0.$$

This equation obviously follows from system (11.4), if $p \in C^2$, $u \in C^2$. This equation is called the *string equation*, since the graph of the function p can be interpreted as a form of small oscillations of a string. The string equation is a special case of the *wave equation*

$$p_{tt} - c^2 \Delta p = 0, \quad p = p(x,t), \quad x \in \mathbb{R}^n, \ t > 0. \tag{11.22}$$

Here, Δ is the Laplace operator. For $n = 2$ the wave equation describes the oscillations of a membrane, for $n = 3$ it describes the oscillations of 3-dimensional medium.

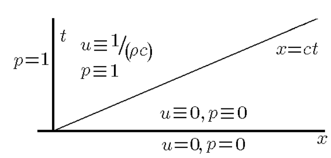
11.18. Remark. The string equation is remarkable in many aspects. It was the first equation in partial derivatives that appeared in mathematical investigations (B. Teylor, 1713). It was a source of fruitful discussion (see, for instance, [**42, 50**]) in which the notion of a function was developed (d'Alembert, Euler, D. Bernoulli, Fourier, Riemann,...).

11.19. Remark. Many distinctive properties of some differential operators $A(y, \partial_{y_1}, \dots \partial_{y_m})$, $y \in \Omega$ are defined by the properties of the corresponding characteristic polynomials $A(y, \eta_1, \dots, \eta_m)$ of the variable $\eta = (\eta_1, \dots, \eta_m)$. Thus, the hyperbolic polynomial $\tau^2 - \xi^2$ $(\tau^2 - |\xi|^2)$ is associated with the string equation $u_{tt} - u_{xx} = 0$ (or, more generally, with the wave equation $u_{tt} - \Delta u = 0$); the elliptic polynomial $|\xi|^2 \equiv \sum\limits_k \xi_k^2$ is associated with the Laplace equation $\Delta u = 0$; the parabolic polynomial $\tau - |\xi|^2$ is associated with the heat equation $u_t - \Delta u = 0$. According to the type of the characteristic polynomial, partial differential equations can be classified into hyperbolic equations, elliptic equations, parabolic equation, ... (See exact definitions, for instance, in [**48**].)

11.20. Example. Consider problem (11.16)–(11.18) with $f = g = 0$, $h = 1$. This means that at the initial instant $t = 0$ the velocity u and the pressure p are equal to zero, and on the boundary $x = 0$ the pressure $p = 1$ is maintained. Formulae (11.5), (11.20)–(11.21) give the following result:

$$\left.\begin{array}{lll} u = 0, & p = 0, & \text{if } \ t < x/c \\ u = 1/\rho c, & p = 1, & \text{if } \ t \geq x/c \end{array}\right\} \tag{11.23}$$

The functions u and p are discontinuous that is not surprising, because condition (11.9) does not hold. However, on the other hand, formulae (11.23) are in a good accordance with the physical processes. This yields



11.21.**P.** *Find the appropriate definitions of the generalized solutions for the following problems:*

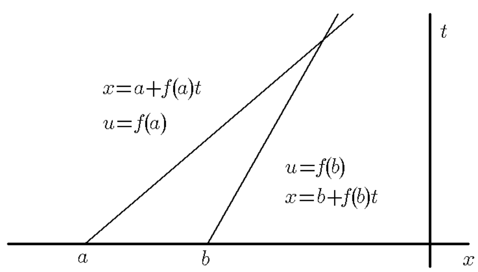
(1) $u_t + u_x = F(x,t)$ *in* $Q = \{(x,t) \in \mathbb{R}^2 \mid x > 0, t > 0\}$*;*
$u\big|_{t=0} = f(x),\ x > 0;\ u\big|_{x=0} = h(t),\ t > 0$*;*
(2) $u_t + (1/\rho)p_x = F(x,t),\ p_t + \rho \cdot c^2 u_x = G(x,t)$ *in* Q*;*
$u\big|_{t=0} = f(x),\ p\big|_{t=0} = g(x),\ x > 0;\ u\big|_{x=0} = h(t),\ t > 0.$
(3) $p_{tt} - c^2 p_{xx} = F(x,t)$ *in the half-strip* $\Omega = \{(x,t) \in \mathbb{R}^2_+ \mid 0 < x < 1\}$*;*
$p\big|_{t=0} = f(x),\ p_t\big|_{t=0} = g(x),\ 0 < x < 1;\ \ p\big|_{x=0} = h_0(x)$,
$p\big|_{x=1} = h_1(x),\ t > 0.$

Revise the requirements to the functions f, g, h, F, G, under which the solutions of these problems belong, say, to the space C^1, PC^1 or L^1_{loc}. Prove the theorems of existence, uniqueness and continuous dependence (compare with P.11.11).

We conclude this section by consideration of the non-linear equation

$$u_t + (u^2/2)_x = \epsilon u_{xx} + b_x(x,t), \quad u = u(x,t), \tag{11.24}$$

where $\epsilon \geq 0$, and b is a given function. This equation is called the *Burgers equation* and is considered in hydrodynamics as a model equation for $\epsilon > 0$ of the Navier–Stokes system, and for $\epsilon = 0$ of the Euler system (see [**52**]).



First, consider equation (11.24) for $\epsilon = 0$ and $b \equiv 0$. In this case, the regular (of the class C^1) solution of this equation satisfies the system $dx/dt = u$, $du/dt = 0$. Thus, the solution $u(x,t)$ is constant along the *characteristic*, i.e., along the curve defined by the equation $dx/dt = u(x,t)$; hence, this curve is, in fact, the straight line $x = a + f(a)t$ that depends only on the parameter $a \in \mathbb{R}$ and a function f. The function f is determined by the relations $f(a) = dx/dt$, $dx/dt = u(x,t)$, i.e., $f(x) = u(x,0)$. If f is a decreasing function, for instance, $f(x) = -\operatorname{th}(x)$, then the characteristics intersect at some $t > 0$, and at the point of the intersection we have

$$u(x-0,t) > u(x+0,t). \tag{11.25}$$

The continuous solution cease to exist. Thus, the Cauchy problem

$$u_t + (u^2/2)_x = 0 \quad \text{in } \mathbb{R}^2_+, \quad u\big|_{t=0} = f(x), \quad x \in \mathbb{R} \tag{11.26}$$

has, in general, no continuous solution even for analytic initial data. This effect is well known in hydrodynamics. It is connected with arising of the so-called *shock waves*[5)] which are characterized by a jump-like change of the density, velocity, etc. Thus, physics suggest that the solution of problem (11.26) should be sought as a generalized solution of the class PC^1.

[5)] See Addendum.

Suppose that u is a generalized solution of problem (11.26), and u has a jump along the curve

$$\gamma = \{(x,t) \in \mathbb{R}^2 \mid x = \lambda(t),\ \lambda \in C^1[\alpha,\beta]\};$$

more exactly, suppose that $(x,t) \in \gamma$ satisfies condition (11.25).

11.22.**P.** *Prove (compare with P.12.6) that the Hugoniót condition*

$$d\lambda(t)/dt = [u(\lambda(t)+0,t) + u(\lambda(t)-0,t)]/2. \tag{11.27}$$

holds along this line γ called the break line.

One can show (see, for instance, [**52**]) that relations (11.25), (11.27) replace the differential equation $u_t + (u^2/2)_x = 0$ on the break line.

One of the approaches to the study of problem (11.26) is based on consideration of the Cauchy problem for equation (11.24) for $\epsilon > 0$ (and $b \equiv 0$) with the passage to the limit as $\epsilon \to 0$ (see, for instance, [**52**]). The point is that (for $b \equiv 0$) equation (11.24) can be reduced, however surprising it is, to the well studied heat equation. Actually, the following theorem holds[6)].

[6)] This theorem was proven in 1948 by V.A. Florin and in 1950 was rediscovered by E. Hopf.

11.23. Theorem. *The solution of equation* (11.24) *can be represented in the form $u = -2\epsilon(\ln G)_x$, where G is the solution of the* *linear parabolic equation*

$$G_t = \epsilon G_{xx} - \frac{b(x,t)}{2\epsilon} G. \tag{11.28}$$

PROOF. Let u be the solution. Setting

$$P(x,t) = u(x,t), \quad Q(x,t) = -u^2(x,t)/2 + \epsilon u_x(x,t) + b(x,t),$$

we have $P_t = u_t$, $Q_x = -u \cdot u_x + \epsilon u_{xx} + b_x(x,t)$. Therefore, $P_t = Q_x$. Thus, the function is defined

$$F(x,t) = \int\limits_{(0,0)}^{(x,t)} P dx + Q dt.$$

We have $F_x = P$, $F_t = Q$. Hence, $F_t + (F_x)^2/2 - \epsilon F_{xx} = b$. Introducing the function $G = \exp[-F/2\epsilon]$, we obtain that G is the solution of equation (11.28) and $u = -2\epsilon(\ln G)_x$, because $u = F_x$. $\square$

CHAPTER 2

The spaces $\mathcal{D}^\flat$, $\mathcal{D}^\#$ and $\mathcal{D}'$. Elements of the distribution theory (generalized function in the sense of L. Schwartz)

12. The space $\mathcal{D}^\flat$ of the Sobolev derivatives

The definition of the generalized solution $u \in L^1_{loc}$ to one or another problem of mathematical physics given by Sobolev [**61**] and, in particular, Definition 11.7 is based on Theorem 10.1 and the Ostrogradsky–Gauss formula (7.2). Let us recall that Theorem 10.1 asserts the equivalence of the two following representations of an element $u \in L^1_{loc}$:

$$1)\ \Omega \ni x \longmapsto u(x) \quad \text{and} \quad 2)\ C_0^\infty(\Omega) \ni \varphi \longmapsto \int\limits_\Omega u(x)\varphi(x)dx,$$

and formula (7.2) implies that, for the differential operator $\partial^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}$ and any function $u \in C^{|\alpha|}(\Omega)$ the following identity is valid:

$$\int\limits_\Omega (\partial^\alpha u(x))\varphi(x)dx = (-1)^{|\alpha|}\int\limits_\Omega u(x)\partial^\alpha\varphi(x)dx \quad \forall\varphi \in C_0^\infty(\Omega).$$

Thus, the functional

$$\partial^\alpha u : C_0^\infty(\Omega) \ni \varphi \longmapsto \langle\partial^\alpha u, \varphi\rangle = (-1)^{|\alpha|}\int\limits_\Omega u(x)\partial^\alpha\varphi(x)dx \qquad \forall\varphi \in C_0^\infty(\Omega) \quad (12.1)$$

determines the function $\partial^\alpha u(x)$, if $u \in C^{|\alpha|}(\Omega)$. Since functional (12.1) is also defined for $u \in L^1_{loc}(\Omega)$, one can, tracing S.L. Sobolev's approach, give the following definition.

12.1. DEFINITION. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multiindex. The weak derivative of order α of the function $u \in L^1_{loc}(\Omega)$ is defined as the functional $\partial^\alpha u$ given by formula (12.1).

Using Theorem 10.1, formula (7.2), and the identity

$$\int_\Omega a(x)(\partial^\alpha u(x))\varphi(x)\, dx = (-1)^{|\alpha|} \int_\Omega u(x)\partial^\alpha(a(x)\varphi(x))\, dx,$$
$$u \in C^{|\alpha|}(\Omega), \quad \varphi \in C_0^\infty(\Omega),$$

which is valid for any function $a \in C^\infty(\Omega)$, we introduce the operation of multiplication of the functional $\partial^\alpha u$, where $u \in L^1_{loc}(\Omega)$, by a function $a \in C^\infty(\Omega)$ with the help of the formula

$$a\partial^\alpha u : C_0^\infty(\Omega) \ni \varphi \longmapsto (-1)^{|\alpha|} \int_\Omega u(x)\partial^\alpha(a(x)\varphi(x))dx \in \mathbb{C}. \quad (12.2)$$

12.2. DEFINITION. The *space of Sobolev derivatives* is the space of functionals of the form $\sum\limits_{|\alpha|<\infty} \partial^\alpha u_\alpha$, where α is a multiindex and $u \in L^1_{loc}(\Omega)$, equipped with the operation of multiplication by formula (12.2). This space is denoted by $\mathcal{D}^\flat(\Omega)$.

12.3. EXAMPLE. Let the function $x_+ \in L^1_{loc}(\mathbb{R})$ be defined in the following way: $x_+ = x$ for $x > 0$, $x_+ = 0$ for $x < 0$. Let us find its derivatives. We have

$$\langle x'_+, \varphi\rangle = -\langle x_+, \varphi'\rangle = -\int_{\mathbb{R}} x_+\varphi'(x)dx = -\int_{\mathbb{R}_+} x\varphi'(x)dx$$
$$= -x\varphi(x)\big|_0^\infty + \int_0^\infty \varphi(x)dx = \int_{\mathbb{R}} \theta(x)\varphi(x)dx = \langle \theta, \varphi\rangle,$$

i.e., $x'_+ = \Theta$ is the Heaviside function. Now find x''_+, i.e., Θ'. We have

$$\langle \Theta', \varphi\rangle = -\langle \Theta, \varphi'\rangle = -\int_0^\infty \varphi'(x)dx = -\varphi(x)\big|_0^\infty = \varphi(0) = \langle \delta, \varphi\rangle, \quad (12.3)$$

i.e., $\Theta' = \delta(x)$ is the Dirac δ-function. In the same way one can find any derivative of the δ-function of order k. We have

$$\langle \delta^{(k)}, \varphi\rangle = -\langle \delta^{(k-1)}, \varphi'\rangle = \cdots = (-1)^k \langle \delta, \varphi^{(k)}\rangle = (-1)^k \varphi^{(k)}(0). \tag{12.4}$$

12.4. **P.** *Let* $\Theta_\epsilon \in C^\infty(\mathbb{R})$, $0 \le \Theta_\epsilon(x) \le 1$, *and* $\Theta_\epsilon(x) \equiv 1$ *for* $x > \epsilon$ *and* $\Theta_\epsilon(x) \equiv 0$ *for* $x < -\epsilon$. *Let us set* $\delta_\epsilon(x) = \Theta'_\epsilon(x)$. *Show that* $\lim_{\epsilon\to 0}\langle \delta_\epsilon^{(k)}, \varphi\rangle = (-1)^k \varphi^{(k)}(0)$ $\forall k \ge 0$, $\forall \varphi \in C_0^\infty(\mathbb{R})$.

12.5. Remark. Formulae (12.4) allow us to extend the functional $\delta^{(k)}$ from the functional space $C_0^\infty(\mathbb{R})$ to the space of functions k-times continuously differentiable at the point $x = 0$ (see Definition 2.2). On the other hand, formula (12.3) is not defined on the space $C(\mathbb{R})$, because the functional Θ is not defined on $C(\mathbb{R})$.

Define the function $\Theta_\pm : \mathbb{R}^n \ni x \mapsto \Theta_\pm(x)$ by the following formula

$$\begin{aligned} &\Theta_\pm(x) = 1_{Q_\pm}(x),\ x \in \mathbb{R}, \\ &\quad Q_\pm = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid \pm x_k > 0\ \forall k\}. \end{aligned} \tag{12.5}$$

If $n = 1$, then $\Theta_+ = \Theta$ is the Heaviside function, and $\Theta_- = 1 - \Theta_+$ (in L^1_{loc}).

12.6. **P.** *(cf. P.11.22). Let* $F \in C^1(\mathbb{R})$, $\lambda \in C^1(\mathbb{R})$, $u_\pm \in C^1(\mathbb{R}^2)$. *Let, for* $(x,t) \in \Omega \subset \mathbb{R}^2$, $u(x,t) = u_+(x,t)\Theta_+(x-\lambda(t)) + u_-(x,t)\Theta_-(x-\lambda(t))$. *Find* u_t *and* $(F(u))_x$, *noting that* $F(u(x,t)) = F(u_+(x,t))\Theta_+(x-\lambda(t)) + F(u_-(x,t))\Theta_-(x-\lambda(t))$. *Show that* $u_t + (F(u))_x = 0$ *almost everywhere in* Ω *if and only if, first,* $u_t + (F(u))_x \equiv 0$ *in* $\Omega \setminus \gamma$, *where* $\gamma = \{(x,t) \in \mathbb{R}^2 \mid x = \lambda(t)\}$, *and second, the Hugoniót condition* $\frac{d\lambda(t)}{dt} = \frac{F(u(x+0,t) - F(u(x-0,t)}{u(x+0,t)-u(x-0,t)}$ *holds on the line* γ.

12.7. **P.** *Check (see* (12.5)*) that* $\frac{\partial^n}{\partial x_1 \dots \partial x_n}\Theta_+ = \delta(x)$, $x \in \mathbb{R}^n$.

12.8. **P.** *Show that the function* $E(x,t) = \Theta(t - |x|)/2$ *is the fundamental solution of the string operator, i.e.,*

$$(\partial^2/\partial t^2 - \partial^2/\partial x^2)E(x,t) = \delta(x,t).$$

Here, $\delta(x,t)$ *is that* δ*-function in* $\mathbb{R}\times\mathbb{R}$, *i.e.,* $\langle \delta(x,t), \varphi\rangle = \varphi(0,0)$ $\forall \varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R})$.

12.9.**P.** *Noting that, for $\varphi \in C_0^\infty(\mathbb{R})$,*

$$\lim_{\epsilon\to 0} \int\limits_{|x|>\epsilon} \ln|x| \cdot \varphi'(x)dx = \lim_{\epsilon\to 0} \left[\ln\epsilon(\varphi(-\epsilon) - \varphi(\epsilon)) - \int\limits_{|x|>\epsilon} \frac{\varphi(x)}{x} dx \right],$$

prove that $\frac{d}{dx}\ln|x| = v.p.\frac{1}{x}$, i.e., $\left\langle \frac{d}{dx}\ln|x|, \varphi\right\rangle = v.p.\int_{-\infty}^{\infty} \frac{\varphi(x)}{x}dx$ $\forall\varphi \in C_0^\infty(\mathbb{R})$, where $v.p.\int_{-\infty}^{\infty} x^{-1}\varphi(x)\,dx$ is so-called the principal value = valeur principal (French) of the integral $\int_{-\infty}^{\infty} x^{-1}\varphi(x)dx$ defined by the formula

$$v.p. \int\limits_{-\infty}^{\infty} x^{-1}\varphi(x)dx = \lim_{\epsilon\to 0} \int\limits_{|x|>\epsilon} x^{-1}\varphi(x)dx. \tag{12.6}$$

12.10. **P.** *Taking into account that $\ln(x \pm i\epsilon) = \ln|x \pm i\epsilon| + i\arg(x \pm i\epsilon) \to \ln|x| \pm i\pi\Theta(-x)$ as $\epsilon \to +0$, prove the simplest version the Sokhotsky formulae (very widespread in the mathematical physics (see, for instance,* [**38**]*))*

$$\frac{1}{x \mp i0} = v.p.\frac{1}{x} \pm i\pi\delta(x), \tag{12.7}$$

i.e., prove that $\lim\limits_{\epsilon\to+0} \int_{-\infty}^{\infty} \frac{\varphi(x)dx}{x\mp i\epsilon} = v.p.\int_{-\infty}^{\infty} \frac{\varphi(x)}{x}dx \pm i\pi\varphi(0)$ $\forall\varphi \in C_0^\infty(\mathbb{R})$.

12.11. Remark. Formulae (12.7) imply that

$$\delta(x) = f(x - i0) - f(x + i0), \quad \text{where} \quad f(x + iy) = \frac{1}{2\pi i}(x + iy)^{-1}, \tag{12.8}$$

i.e., the δ-function being an element of $\mathcal{D}^\flat(\mathbb{R})$ admits the representation in the form of the difference of boundary values on the real axis of two functions analytic in $\mathbb{C}_+$ and in $\mathbb{C}_-$, respectively, where $\mathbb{C}_\pm = \{z = x + iy \in \mathbb{C} \mid \pm y > 0\}$. This simple observation has deep generalizations in the theory of hyperfunctions (see, for instance, [**43, 53**]).

12.12. Remark. Any continuous function $F \in C(\mathbb{R})$ has, as an element of the space L^1_{loc}, the Sobolev derivative $F' \in \mathcal{D}^\flat(\mathbb{R})$. If this derivative is a locally integrable function, in other words, if

$F(x) = \int_a^x f(y)\,dy + F(a)$, where $f \in L^1_{loc}(\mathbb{R})$, then Theorem 8.27 implies that

$$F'(x) = \lim_{\sigma\to 0} \sigma^{-1}(F(x+\sigma) - F(x)) \text{ for almost all } x \in \mathbb{R}. \quad (12.9)$$

In this case, formula (12.9) totally determines the Sobolev derivative F'. Emphasize that the last assertion does not hold (even under the assumption that formula (12.9) is valid), if $F' \notin L^1_{loc}(\mathbb{R})$. Thus, for instance, the *Cantor ladder* (see [**36**] or [**56**]) corresponding to the Cantor set of zero measure (see hint to P.8.23), i.e., a continuous monotone function $K \in C[0,1]$ with the value $(2k-1)\cdot 2^{-n}$ in kth $(k = 1,\dots,2^{n-1})$ interval $I_n =]a_n^k, b_n^k[$ of rank n (see hint to P.8.23) has, for almost all $x \in [0,1]$, zero derivative but its Sobolev derivative K' is non-zero. Namely,

$$K' = \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} (2k-1)\cdot 2^{-n}(\delta(x-b_n^k) - \delta(x-a_n^k)). \quad (12.10)$$

12.13.**P.** *Prove formula* (12.10).

13. The space $\mathcal{D}^{\#}$ of generalized functions

The elements of the space $\mathcal{D}^\flat$ were defined as finite linear combinations of the functionals $\partial^\alpha u_\alpha$ (12.1), i.e., of the derivatives of the functions $u_\alpha \in L^1_{loc}$. If we neglect the concrete form of the functionals, i.e., consider an arbitrary linear functional

$$f : C_0^\infty(\Omega) \ni \varphi \longmapsto \langle f, \varphi\rangle \in \mathbb{C}, \quad (13.1)$$

then we obtain an element of the space $\mathcal{D}^{\#}(\Omega)$, which will be called a *generalized function* (in the domain Ω). Let us give the exact

13.1. Definition. $\mathcal{D}^{\#}(\Omega)$ is the space of all linear functionals (13.1) in which the operations of differentiation ∂^α and multiplication by a function $a \in C^\infty(\Omega)$ are introduced by the following formulae:

$$\langle \partial^\alpha f, \varphi\rangle = (-1)^{|\alpha|}\langle f, \partial^\alpha\varphi\rangle, \quad \langle af, \varphi\rangle = \langle f, a\varphi\rangle \ \forall\varphi \in C_0^\infty(\Omega). \quad (13.2)$$

13.2. Example. $f = \sum\limits_{k=0}^{\infty} \delta^{(k)}(x-k)$, $x \in \mathbb{R}$, i.e., $\langle f, \varphi\rangle = \sum\limits_{k=0}^{\infty} (-1)^k \varphi^{(k)}(k)$ $\forall \varphi \in C_0^\infty(\mathbb{R})$. Obviously, $f \in \mathcal{D}^{\#}(\mathbb{R})$, and $f \notin$

$\mathcal{D}^\flat(\mathbb{R})$. Thus, $\mathcal{D}^\flat(\Omega) \subsetneq \mathcal{D}^\#(\Omega)$. By the way, the following lemma is valid.

13.3. Lemma (P. du Bois Reimond). *If $f \in \mathcal{D}^\#(\mathbb{R})$ and $f' = 0$, then $f = \text{const}$. (Thus, $f \in \mathcal{D}^\flat(\mathbb{R})$.)*

Proof. We have $\langle f', \varphi\rangle = \langle f, \varphi'\rangle = 0\ \forall \varphi \in C_0^\infty(\mathbb{R})$. Let us take a function $\varphi_0 \in C_0^\infty(\mathbb{R})$ such that $\int \varphi_0 = 1$. Any function $\varphi \in C_0^\infty(\mathbb{R})$ can be represented in the form $\varphi = \varphi_1 + \left(\int \varphi\right)\varphi_0$, where $\varphi_1 = \varphi - \left(\int\varphi\right)\varphi_0$. Note that $\int\varphi_1 = 0$. Setting $\psi(x) = \int_{-\infty}^x \varphi_1(\xi)d\xi$, we have $\psi \in C_0^\infty(\mathbb{R})$ and $\psi' = \varphi_1$. Therefore, $\langle f, \varphi\rangle = \langle f, \psi'\rangle + \langle f, \left(\int\varphi\right)\varphi_0\rangle$. Since $\langle f, \psi'\rangle = 0$, it follows that $\langle f, \varphi\rangle = C\int\varphi$, where $C = \langle f, \varphi_0\rangle$. $\square$

Generalizing the notion of a δ-sequence, we introduce

13.4. Definition. A sequence of functionals $f_\nu \in \mathcal{D}^\#$ is said *weakly converges* to $f \in \mathcal{D}^\#$ on the space $\Phi \supset C_0^\infty$, if $f_\nu \longrightarrow f$ in $\mathcal{D}^\#$ on the space Φ, i.e., $\lim_{\nu\to\infty}\langle f_\nu, \varphi\rangle = \langle f, \varphi\rangle\ \forall\varphi \in \Phi$. If $\Phi = C_0^\infty$, then the words "on the space C_0^∞" are usually omitted.

13.5. Definition. We say that a subspace X of the space $\mathcal{D}^\#$ is complete with respect to the weak convergence, if, for any sequence $\{f_\nu\}_{\nu=1}^\infty$ of functionals $f_\nu \in X$ satisfying the condition

$$\langle f_\nu - f_\mu, \varphi\rangle \longrightarrow 0 \quad \forall\varphi \in C_0^\infty \quad \text{as} \quad \nu,\mu \longrightarrow \infty,$$

there exists $f \in X$ such that $f_\nu \to f$ in $\mathcal{D}^\#$.

13.6. **P.** *Show that $\mathcal{D}^\flat$ is not complete with respect to the weak convergence.*

13.7. **P.** *Show that $\mathcal{D}^\#$ is complete with respect to the weak convergence.*

13.8. Lemma. *If $f_\nu \to f$ in $\mathcal{D}^\#$ on the space $\Phi \supset C_0^\infty$, then $\partial^\alpha f_\nu \to \partial^\alpha f$ in $\mathcal{D}^\#$ on the space Φ for any α.*

Proof. $\langle \partial^\alpha f_\nu, \varphi\rangle = (-1)^{|\alpha|}\langle f_\nu, \partial^\alpha\varphi\rangle \longrightarrow (-1)^{|\alpha|}\langle f, \partial^\alpha\varphi\rangle = \langle\partial^\alpha f, \varphi\rangle$. $\square$

13.9. Example. Let $f_\nu = \frac{\sin\nu x}{\nu}$, i.e., $\langle f_\nu, \varphi\rangle = \int_{\mathbb{R}} \frac{\sin\nu x}{\nu}\varphi(x)dx$. Then $f_\nu' = \cos\nu x$, $f_\nu'' = -\nu\cdot\sin\nu x, \ldots$ We have $\langle f_\nu, \varphi\rangle \longrightarrow 0$ $\forall\varphi \in C_0^\infty$ as $\nu\to\infty$. Thus, $\cos\nu x \to 0$ in $\mathcal{D}^\#$, $\nu\sin\nu x \to 0$ in $\mathcal{D}^\#, \ldots$

13.10. LEMMA. *Let $a = (a_1, \dots, a_n) \in \Omega \subset \mathbb{R}^n$. Suppose that a sequence $\{f_\nu\}_{\nu=1}^{\infty}$ of functions $f_\nu \in L^1_{loc}(\Omega)$ and a point $b = (b_1, \dots, b_n)$*[1] *$\in \Pi$, where*

$$\Pi = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid |x_k - a_k| < \sigma_k,\ \sigma_k > 0\ \forall k\} \subset \Omega,$$

are such that, for $F_\nu(x) = \int_{b_1}^{x_1} \dots \int_{b_n}^{x_n} f_\nu(y)dy_1 \dots dy_n$, the following two properties hold:

[1] *As $b = (b_1, \dots, b_n)$ one can take any point of Π such that $b_k < a_k$ $\forall k$.*

(1) *$|F_\nu(x)| \le G(x)$, $x \in \Omega$, where $G \in L^1_{loc}(\Omega)$,*
(2) *$F_\nu(x) \to \Theta_+(x - a)$ almost everywhere in Ω, where Θ_+ is defined in* (12.5).

Then f_ν weakly converges to $\delta(x - a)$ on the space

$$\Phi = \{\varphi \in C(\Omega) \mid \varphi \in L^1(\Omega),\ \partial^n\varphi/\partial x_1 \dots \partial x_n \in L^1(\Omega)\}. \quad (13.3)$$

PROOF. Using Theorems 8.20, 8.24, and 8.27, we obtain, for any $\varphi \in \Phi$,

$$\begin{aligned}\langle f_\nu, \varphi\rangle &= \left\langle \frac{\partial^n F_\nu}{\partial x_1 \dots \partial x_n}, \varphi \right\rangle = (-1)^n \left\langle F_\nu, \frac{\partial^n \varphi}{\partial x_1 \dots \partial x_n} \right\rangle \\ &= (-1)^n \int\limits_\Omega F_\nu(x) \frac{\partial^n\varphi(x)}{\partial x_1 \dots \partial x_n} dx \longrightarrow (-1)^n \int\limits_{a_1}^{\infty} \dots \int\limits_{a_n}^{\infty} \frac{\partial^n\varphi(x)dx}{\partial x_1 \dots \partial x_n} \\ &= -(-1)^n \int\limits_{a_2}^{\infty} \dots \int\limits_{a_n}^{\infty} \frac{\partial^{n-1}\varphi(x)dx}{\partial x_2 \dots \partial x_n} dx_2 \dots dx_n = \varphi(a).\end{aligned}$$

□

13.11.**P.** *Using Lemma 13.10, solve problems P.4.3 and P.4.4.*

Let us generalize the notion of the support of a function (see Section 3), assuming, in particular, an exact meaning to the phrase usual for physicists: “$\delta(x) = 0$ for $x \neq 0$”.

13.12. DEFINITION. Let $f \in \mathcal{D}^{\#}(\Omega)$, and ω be an open set in Ω. We say that f *is zero (vanishes) on* ω (and write $f\big|_\omega = 0$ or $f(x) = 0$ for $x \in \Omega$), if $\langle f, \varphi\rangle = 0\ \forall \varphi \in C_0^\infty(\omega)$.

13.13. Definition. The *annihilating set of a functional* $f \in \mathcal{D}^\#(\Omega)$ is the maximal open set $\Omega_0 = \Omega_0(f) \subset \Omega$ on which f is zero, i.e., $f\big|_{\Omega_0} = 0$, and the condition $f\big|_\omega = 0$ implies $\omega \subset \Omega_0$.

It is clear that $\Omega_0(f)$ is the union of $\omega \subset \Omega$ such that $f\big|_\omega = 0$.

13.14. Definition. Let $f \in \mathcal{D}^\#(\Omega)$. The *support* of the functional f, denoted by $\operatorname{supp} f$, is the completion to the annihilating set $\Omega_0(f)$, i.e., the set $\Omega \setminus \Omega_0(f)$.

13.15.**P.** *Let $f \in \mathcal{D}^\#(\Omega)$. Check that $x \in \operatorname{supp} f$ if and only if, for any neighbourhood $\omega \subset \Omega$ of the point x there exists a function $\varphi \in C_0^\infty(\omega)$ such that $\langle f, \varphi\rangle \neq 0$. Verify also that Definition 13.14 is equivalent to Definition 3.1.4, if $f \in C(\Omega)$.*

13.16.**P.** *Find* $\operatorname{supp} \delta^{(\alpha)}(x)$ *and* $\operatorname{supp}[(x_1 + \cdots + x_n)\delta^{(\alpha)}(x)]$.

13.17.**P.** *Let $f \in \mathcal{D}^\#(\Omega)$, $a \in C^\infty$, and $a(x) = 1$ for $x \in \operatorname{supp} f$. Is it true that $a \cdot f = f$?*

13.18.**P.** *Let ω be an open set in Ω such that $\omega \supset \operatorname{supp} f$, $f \in \mathcal{D}^\#(\Omega)$. Show that $af = f$, if $a(x) = 1$ for $x \in \omega$.*

13.19.**P.** *Let $f \in \mathcal{D}^\#(\Omega)$ be a generalized function with a compact support. Show that the formula $\langle F, \varphi\rangle = \langle f, \psi\varphi\rangle$ $\forall \varphi \in C^\infty(\Omega)$, $\psi \in C_0^\infty(\Omega)$, where $\psi \equiv 1$ on an open set $\omega \supset \operatorname{supp} f$ defines the extension of the functional f onto the space $C^\infty(\Omega)$, i.e., F is a linear functional on $C^\infty(\Omega)$ such that $\langle F, \varphi\rangle = \langle f, \varphi\rangle$ $\forall \varphi \in C_0^\infty(\Omega)$.*

14. The problem of regularization

The idea of representability of a function $f : \Omega \longrightarrow \mathbb{C}$ with the help of its "averaging" functional (10.1) concerned only locally integrable functions. However, in many problems of analysis, an important role is played by functions which are not locally integrable. This is the reason of arising of the so-called *problem of regularization*: let $g : \Omega \ni x \mapsto g(x)$ be a function locally integrable everywhere in Ω except a subset $N \subset \Omega$. It is required to find functionals $f \in \mathcal{D}^\#$ such that

$$\langle f, \varphi\rangle = \int\limits_\Omega g(x)\varphi(x)\,dx \quad \forall \varphi \in C_0^\infty(\Omega \setminus N). \tag{14.1}$$

In this case one says that the functional f regularizes the (divergent) integral $\int_\Omega g(x)\,dx$.

It is clear that the functionals f satisfying (14.1) can be represented in the form

$$f = f_0 + f_1, \quad f_0 \in F_0,$$

where f_1 is a particular solution of the problem of regularization (i.e., f_1 satisfies (14.1)), and F_0 is the linear subspace of the functionals $f_0 \in \mathcal{D}^{\#}(\Omega)$ such that

$$\langle f_0, \varphi \rangle = 0 \quad \forall \varphi \in C_0^\infty(\Omega \setminus N). \tag{14.2}$$

The question of description of the subspace F_0 is connected only with the set $N \supset \operatorname{supp} f_0$. In the case when $N = x_0 \in \Omega$, this question, i.e., the problem concerning the general form of functionals with a point support, is considered in Section 15. As for the particular solution of the problem regularization, we conclude this section by consideration of the regularization for $1/P$, where P is a polynomial in the variable $x \in \mathbb{R}$.

14.1. Example. Consider the regularization of the function $1/x$. In other words, find the functional $f \in \mathcal{D}^{\#}(\mathbb{R})$ which satisfies the condition: $x \cdot f = 1$. Note (see (12.6)) that

$$\left\langle \text{v.p.}\frac{1}{x}, \varphi \right\rangle = \int_{-\infty}^{\infty} \frac{1}{x}\varphi(x)\,dx \quad \forall \varphi \in C_0^\infty(\mathbb{R} \setminus 0).$$

Thus, the functional v.p.$(1/x)$ regularizes the function $1/x$. Since $\langle \delta, \varphi \rangle = 0 \ \forall \varphi \in C_0^\infty(\mathbb{R} \setminus 0)$, it follows that v.p.$(1/x) + C \cdot \delta(x)$, where $C \in \mathbb{C}$; therefore, (see (12.7)) functional $1/(x \pm i0)$ also regularize the function $1/x$.

14.2.**P.** *Check that*

$$\left\langle v.p.\frac{1}{x}, \varphi \right\rangle = \int_{-\infty}^{\infty} \frac{\varphi(x) - \varphi(-x)}{2x}\,dx \quad \forall \varphi \in C_0^\infty(\mathbb{R}).$$

14.3. **P.** *Let $m \geq 1$, and $a \in C_0^\infty(\mathbb{R})$. Define, for $k \geq 2$, the functional v.p.$(1/x^k) \in \mathcal{D}^\#(\mathbb{R})$ by the formulae:*

$$\left\langle v.p.\frac{1}{x^k}, \varphi \right\rangle = \int_0^\infty \frac{1}{x^k}\left\{\varphi(x) + \varphi(-x) - 2\left[\varphi(0) + \cdots + \frac{x^{k-2}}{(k-2)!}\varphi^{(k-2)}(0)\right]\right\}dx$$

for $k = 2m$ and

$$\left\langle v.p.\frac{1}{x^k}, \varphi \right\rangle = \int_0^\infty \frac{1}{x^k}\left\{\varphi(x) + \varphi(-x) - 2\left[x\varphi'(0) + \cdots + \frac{x^{k-2}}{(k-2)!}\varphi^{(k-2)}(0)\right]\right\}dx$$

for $k = 2m + 1$.

Show that the functional v.p.$(1/x^k)$ regularizes the function $1/x^k$.

14.4.**P.** *(Compare with P.16.25). Find the solution $f \in \mathcal{D}^\#(\mathbb{R})$ of the equation $P(x)f = 1$. In other words, regularize the integral $\int_{-\infty}^{\infty} P^{-1}(x)\varphi(x)dx$, where P is a polynomial.*

15. Generalized functions with a point support. The Borel theorem

It has been shown in Section 14 that the mean value problem for a function locally integrable everywhere in $\Omega \subset \mathbb{R}^n$ except a point $\xi \in \Omega$ leads to the question on the general form of the functional $f \in \mathcal{D}^\#(\Omega)$ concentrated at the point ξ, i.e., satisfying the condition: $\operatorname{supp} f = \xi$. It is clear (see P.13.16), that a finite sum of the δ-function and its derivatives concentrated at the point ξ, i.e., the sum

$$\sum_{|\alpha| \leq N} c_\alpha \delta^{(\alpha)}(x - \xi), \quad c_\alpha \in \mathbb{C},\ N \in \mathbb{N} \tag{15.1}$$

is an example of such a functional.

However, is the sum (15.1) the general form of a functional $f \in \mathcal{D}^\#$, whose support is concentrated at the point ξ? One can show that the answer to this question is negative, however, the following theorem holds.

15.1. THEOREM. *If* $f \in \mathcal{D}^{\#}$ *and* $f = \sum_{\alpha} c_\alpha \delta^{(\alpha)}(x-\xi)$, *then* $c_\alpha = 0$ *for* $|\alpha| > N_f$ *for some* N_f.

PROOF. According to the Borel theorem below, there exists a function $\varphi \in C_0^\infty(\Omega)$ such that, for any α

$$\partial^\alpha \varphi(x)\big|_{x=\xi} = (-1)^{|\alpha|}/c_\alpha, \quad \text{if} \;\; c_\alpha \neq 0$$

and

$$\partial^\alpha \varphi(x)\big|_{x=\xi} = 0, \quad \text{if} \;\; c_\alpha = 0.$$

For such a function φ, we have $\left\langle \sum_{\alpha} c_\alpha \delta^{(\alpha)}(x-\xi), \varphi \right\rangle = \sum_{\alpha} 1$, where the sum is taken over α for which $c_\alpha \neq 0$. $\square$

15.2. THEOREM (E. Borel). *For any set of numbers* $a_\alpha \in \mathbb{C}$, *parametrized by the multiindices* $\alpha = (\alpha_1, \ldots, \alpha_n)$, *and for any point* $\xi \in \Omega \subset \mathbb{R}^n$, *there exists a function* $\varphi \in C_0^\infty(\Omega)$ *such that* $\partial^\alpha \varphi\big|_{x=\xi} = a_\alpha \; \forall \alpha$.

PROOF. Without loss of generality, we can assume that $\xi = 0 \in \Omega$. If the coefficients a_α grow not very fast as $|\alpha| \to \infty$, more exactly, if there exist $M > 0$ and $\rho > 0$ such that $\sum_{|\alpha|=k} a_\alpha \leq M\rho^{-k} \; \forall k \in \mathbb{N}$, then the existence of the function required is obvious. Actually, since in the case considered the series $\sum_{\alpha} a_\alpha x^\alpha/\alpha!$, where $\alpha! = \alpha_1! \cdot \cdots \cdot \alpha_n!$, converges in the ball $B_\rho = \{x \in \mathbb{R}^n \mid |x| < \rho\}$, we can take as the required function the following one

$$\varphi(x) = \psi(x/\rho) \sum_{\alpha} a_\alpha x^\alpha/\alpha! \in C_0^\infty(B_\rho) \subset C_0^\infty(\Omega),$$

where

$$\psi \in C_0^\infty(\mathbb{R}^n), \quad \psi = 0 \;\; \text{for} \;\; |x| > 1, \psi = 1 \;\; \text{for} \;\; |x| < 1/2.$$

However, in the general case, the series $\sum_{\alpha} a_\alpha x^\alpha/\alpha!$ can diverge in B_ρ. What is the reason of the divergence? Obviously, because it is impossible to guarantee the sufficiently fast decrease of $a_\alpha x^\alpha/\alpha!$ as $|\alpha| \to \infty$ for all x belonging to a fixed ball B_ρ. One can try to improve the situation, by considering the series

$$\sum_{\alpha} \psi(x/\rho_\alpha) \cdot a_\alpha x^\alpha/\alpha!, \tag{15.2}$$

where ρ_α converges sufficiently fast to zero as $|\alpha| \to \infty$. If it occurs that series (15.2) converges to a function $\varphi \in C^\infty$, then, as one can easily see, $\varphi \in C_0^\infty(\Omega)$ and $\partial^\alpha \varphi\big|_{x=0} = a_\alpha$. Indeed, setting $\gamma = (\gamma_1, \ldots, \gamma_n) \le \beta = (\beta_1, \ldots, \beta_n)$ by definition, if $\gamma_k \le \beta_k$ $\forall k$, and $\beta - \gamma = (\beta_1 - \gamma_1, \ldots, \beta_n - \gamma_n)$, we have

$$\begin{aligned}
\partial^\alpha \varphi\big|_{x=0} &= \sum_\beta (a_\beta/\beta!) \left(\sum_{\gamma \le \alpha} \frac{\alpha!}{\gamma!(\alpha-\gamma)!} \left(\partial^{a-\gamma}\psi\right)\big|_{x=0} \left(\partial^\gamma x^\beta\right)\big|_{x=0} \right) \\
&= \sum_\beta (a_\beta/\beta!) \left(\partial^\alpha x^\beta\big|_{x=0}\right) \\
&= \sum_{\beta \ne \alpha} (a_\beta/\beta!) \left(\partial^\alpha x^\beta\big|_{x=0}\right) + a_\alpha = a_\alpha.
\end{aligned}$$

It remains to show that series (15.2) converges to $\varphi \in C^\infty(\Omega)$. Note that since $\sum\limits_\alpha = \sum\limits_{|\alpha| \le k} + \sum\limits_{|\alpha| > k}$, it is sufficient to verify that there exist numbers $\rho_\alpha < 1$ such that

$$\sum_{j>k} \sum_{|\alpha|=j} \psi(x/\rho_\alpha) a_\alpha x^\alpha/\alpha! \in C^k(\Omega) \quad \forall k.$$

Let us try to find $\rho_\alpha = \rho_j$ depending only on $j = |\alpha|$. If we can establish that $\forall \beta$ such that $|\beta| \le k$, the following inequality holds

$$\left|\partial^\beta(\psi(x/\rho_{|\alpha|}) a_\alpha x^\alpha/\alpha!\right| \le C_\alpha \rho_\alpha, \tag{15.3}$$

where $C_\alpha = C_\alpha(\psi) < \infty$, then, taking $\rho_j = 2^{-j}\left(\sum\limits_{|\alpha|=j} C_\alpha\right)^{-1}$, we obtain

$$\sum_{j>k} \sum_{|\alpha|=j} \left|\partial^\beta(\psi(x/\rho_{|\alpha|}) a_\alpha x^\alpha/\alpha!\right| \le \sum_{j>k} \left[\rho_j \sum_{|\alpha|=j} C_\alpha\right] \le 1.$$

Thus, it remains to prove (15.3). For $|\alpha| > k \geq |\beta|$, we have

$$
\begin{aligned}
\left|\partial_x^\beta(\psi\left(\frac{x}{\rho_{|\alpha|}}\right)\frac{a_\alpha x^\alpha}{\alpha!})\right| \leq & \frac{|a_\alpha|}{\alpha!}\sum_{\gamma\leq\beta}\frac{\beta!}{\gamma!(\beta-\gamma)!}\left|\partial_x^\gamma\psi\left(\frac{x}{\rho_{|\alpha|}}\right)\right|\cdot|\partial^{\beta-\gamma}x^\alpha| \\
\leq & \frac{|a_\alpha|x^\alpha}{\alpha!}\sum_{\gamma\leq\beta}\frac{\beta!}{\gamma!(\beta-\alpha)!}\left(\frac{1}{\rho_{|\alpha|}}\right)^{|\gamma|} \\
& \times\left|\partial_t^\gamma\psi(t)\big|_{t=x/\rho_{|\alpha|}}\cdot x^{\alpha-\beta+\gamma}\right|\cdot\alpha \\
\leq & |a_\alpha|\sum_{\gamma\leq\beta}\frac{\beta!}{\gamma!(\beta-\alpha)!}\cdot\left|\partial_t^\gamma\psi(t)\big|_{t=x/\rho_{|\alpha|}}\right|\cdot\rho_{|\alpha|}.
\end{aligned}
$$

□

Now return to the question on the general form of the generalized function $f \in \mathcal{D}^\#(\Omega)$ with the support at the point $\xi = 0 \in \Omega$. First of all, note (see P.13.19) that, for any function $a \in C_0^\infty(\Omega)$ such that $a \equiv 1$ in a neighbourhood of the point $\xi = 0$, the formula $\langle f, \varphi\rangle = \langle f, a\varphi\rangle\ \forall\varphi \in C^\infty(\Omega)$ is valid. In particular, the functional f is defined on the polynomials. Setting $c_\alpha = (-1)^{|\alpha|}\langle f, x^\alpha/\alpha!\rangle$, we obtain

$$\langle f, \varphi\rangle = \sum_{|\alpha|<N} c_\alpha\langle\delta^{(\alpha)}, \varphi\rangle + \langle f, r_N\rangle \quad \forall N,$$

where

$$r_N(x) = a\left(\frac{x}{\epsilon_N}\right)\left[\varphi(x) - \sum_{|\alpha|<N}\varphi^{(\alpha)}(0)x^\alpha/\alpha!\right], \quad 0 < \epsilon_N < 1. \tag{15.4}$$

It is rather tempting to assume that, for an appropriate sequence $\{\epsilon_N\}_{N=1}^\infty$, $0 < \epsilon_N < 1$, the following condition holds:

$$\langle f, r_N\rangle \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty, \tag{15.5}$$

because in this case Theorem 15.1 implies the obvious

15.3. PROPOSITION. *If* $f \in \mathcal{D}^\#(\Omega)$, $\operatorname{supp} f = 0 \in \Omega$ *and* (15.5) *is valid, then* $\exists N \in \mathbb{N}$ *such that* $f = \sum_{|\alpha|\leq N} c_\alpha\delta^{(\alpha)}$.

However, in general, condition (15.5) does not hold, if $f \in \mathcal{D}^\#$. An appropriate example can be constructed with the help of so-called Hamel basis (see, for instance, [**36**]).

16. The space $\mathcal{D}'$ of generalized functions (distributions by L. Schwartz)

The wish seems natural to have a theory of generalized functions in which condition (15.5) is satisfied, hence, Proposition 15.3 holds. This modest wish (leading, as one can see below, to the theory of the Schwartz distributions) suggests the following:

1) introduce a convergence in the space $C_0^\infty(\Omega)$ such that for this convergence
$$\lim_{N\to\infty} r_N = 0 \in C_0^\infty(\Omega), \tag{16.1}$$
where r_N is defined in (15.4) for some suitable $\epsilon_N \in]0,1[$;
2) consider below only the functionals $f \in \mathcal{D}^\#(\Omega)$, which are continuous with respect to the convergence introduced.

It is clear that one can introduce different convergences according to which $r_N \to 0$ as $N \to \infty$. Which one should be chosen? Considering this question, one should take into account that the choice of one or another convergence also determines the subspace of linear functionals on C_0^∞ that are continuous with respect to this convergence. Therefore, it seems advisable to add to items 1) and 2) above the following requirement:

3) the space of functionals continuous with respect to the convergence introduced must include the space $\mathcal{D}^\flat$ of the Sobolev derivatives (since this space, as has been shown, plays very important role in the problems of mathematical physics).

According to Theorem 16.1 below, requirement 3) uniquely determines the convergence in the space C_0^∞; moreover, (see Proposition 16.10) condition (16.1) is also satisfied.

16.1. THEOREM. *Let $\{\varphi_j\}$ be a sequence of functions $\varphi_j \in C_0^\infty(\Omega)$. Then the following two conditions are equivalent:*

1°. $\langle f, \varphi_j\rangle \to 0$ *as* $j \to \infty$ $\forall f \in \mathcal{D}^\flat(\Omega)$;

2°. a) *there exists a compact* $K \subset \Omega$ *such that* $\operatorname{supp}\varphi_j \subset K$ $\forall j$;
b) $\max_{x\in\Omega} |\partial^\alpha \varphi_j(x)| \to 0$ *as* $j \to \infty$ $\forall \alpha$.

PROOF. The implication $2° \implies 1°$ is obvious. The converse assertion follows from Lemmas 16.2–16.5 below. $\square$

16.2. LEMMA. *$\forall\alpha$ $\exists C_\alpha$ such that* $\max\limits_{x\in\Omega}|\varphi_j^{(\alpha)}(x)| \le C_\alpha$ $\forall j$.

PROOF. For any α, consider the sequence of functionals

$$\varphi_j^{(\alpha)} : L^1(\Omega) \ni f \longmapsto \int\limits_\Omega f(x)\partial^\alpha\varphi_j(x)dx, \quad j \ge 1,$$

defined on the space $L^1(\Omega)$. The functionals $\varphi_j^{(\alpha)}$ are, obviously, linear and continuous, i.e., (by the Riesz theorem 9.14) $\varphi_j^{(\alpha)} \in L^\infty$. According to condition 1°, $\langle\varphi_j^{(\alpha)}, f\rangle \to 0$ as $j\to\infty$ $\forall f \in L^1$. Therefore, by virtue of the Banach–Steinhaus theorem[1)] there exists a constant C_α such that $\|\varphi_j^{(\alpha)}\|_\infty \le C_\alpha$ $\forall j$. □

[1)] **The Banach–Steinhaus theorems** (1927) asserts the following (see, for instance, [**36**] or [**56**]). Let X be a Banach space and $\{\varphi_j\}$ be a family of linear continuous functionals on X. If for any $x \in X$ there exists $C_x < \infty$ such that $|\langle\varphi_j, x\rangle| \le C_j$ $\forall j$, then there exists a constant $C < \infty$ such that $|\langle\varphi_j, x\rangle| \le C$ for $\|x\| \le 1$ and $\forall j$.

PROOF. Suppose the contrary be true and note that if a sequence of functionals φ_j is not bounded for $\|x\| \le 1$, then it is not also bounded in the ball $B_r(a) = \{x \in X \mid \|x-a\| \le r\}$. Let us take a point $x_1 \in B_1(0)$, a functional φ_{k_1} and a number $r_1 < 1$ such that $|\langle\varphi_{k_1}, x\rangle| > 1$ for $x \in B_{r_1}(x_1) \subset B_1(0)$. Then we take a point $x_2 \in B_{r_1}(x_1)$, a functional φ_{k_2} and a number $r_2 < r_1$ such that $|\langle\varphi_{k_2}, x\rangle| > 2$ for $x \in B_{r_2}(x) \subset B_{r_1}(x_1)$. Continuing this construction, we obtain a sequence of closed balls $B_{r_k}(x_k)$ embedded in each other, whose radii tend to zero. In this case, $|\langle\varphi_{k_j}, x_0\rangle| > j$ for $x_0 \in \cap B_{r_k}$ (the intersection $\cap B_{r_k}$ is non-empty by virtue of completeness of X). □

16.3. LEMMA. *$\forall\alpha$ $\forall x_0 \in \Omega$ $\partial^\alpha\varphi_j(x_0) \longrightarrow 0$ as $j \longrightarrow \infty$.*

PROOF. $\partial^\alpha\varphi_j(x_0) = \langle\delta^{(\alpha)}(x-x_0), (-1)^{|\alpha|}\varphi_j(x)\rangle \to 0$, since $\delta^{(\alpha)}(x-x_0) \in \mathcal{D}^\flat(\Omega)$. □

16.4. LEMMA. *There exists a compact $K \subset \Omega$ such that for all j we have* $\operatorname{supp}\varphi_j \subset K$.

PROOF. Suppose the contrary is true. Let $K_j = \bigcup\limits_{k<j} \operatorname{supp} \varphi_k$. We can assume that the intersection $(\operatorname{supp} \varphi_j) \cap (\Omega \setminus K)$ is nonempty, i.e., $\exists x_j \in \Omega \setminus K_j$ such that $\varphi_j(x_j) \neq 0$. For any j, we choose $\lambda_j > 0$ such that

$$\frac{|\varphi_j(x)|}{|\varphi_j(x_j)|} > \frac{1}{2} \quad \forall x \in V_j = \{|x - x_j| < \lambda_j\} \subset M_j = \operatorname{supp} \varphi_j \setminus K_j. \tag{16.2}$$

Note that $V_j \cap V_k$ is empty for $j \neq k$ and consider the function $f \in L^1_{loc}(\Omega)$ which is equal to zero outside $\cup_{j \geq 1} V_j$ and such that

$$f(x) = a_j |\varphi_j(x_j)|^{-1} \exp[-i \arg \varphi_j(x)] \quad \text{for} \quad x \in V_j,\ j \geq 1, \tag{16.3}$$

where $a_j > 0$ are constants which will be chosen such that we obtain an inequality contradicting[2)] inequality

$$\left| \int\limits_{\Omega} f \varphi_j dx \right| \geq j. \tag{16.4}$$

Note that $\operatorname{supp} f\varphi_j \subset (V_j \cup K_j)$, because $\operatorname{supp} \varphi_j \subset (M_j \cup K_j)$. Therefore, the last integral in the equality

$$\int\limits_{\Omega} f \varphi_j \, dx = \int\limits_{V_j} \varphi_j \, dx + \int\limits_{(\operatorname{supp} f\varphi_j) \setminus V_j} f \varphi_j \, dx$$

can be estimated by $\left| \int_{K_j} f \varphi_j dx \right| \leq \max\limits_{\Omega} |\varphi_j| \int_{K_j} |f| dx \leq A_j$, where $A_j = C \sum\limits_{k<j} |a_k| \cdot \mu(V_k)$. Taking $a_j = 2(A_j + j)$, we obtain (16.4), since, by virtue of (16.2)–(16.3), $\int_{V_j} f \varphi_j dx \geq a_j/2$. $\square$

[2)] Inequality (16.4) contradicts the initial condition 1°.

16.5. LEMMA. *For any α, $\epsilon > 0$, $x_0 \in \Omega$ there exist λ, $\nu \geq 1$ such that $|\varphi_j^{(\alpha)}(x)| < \epsilon$ for $|x - x_0| < \lambda$ and $j \geq \nu$.*

PROOF. Suppose the contrary is true. Then $\exists \alpha\ \exists \epsilon_0 > 0\ \exists x_0 \in \Omega$ such that, for any j, $\exists x_j \in \{x \in \Omega \mid |x - x_0| < 1/j\}$ such that the inequality $|\varphi_j^{(\alpha)}(x_j)| \geq \epsilon_0$ holds. However, on the other hand,

$$|\varphi_j^{(\alpha)}(x_j)| \leq |\varphi_j^{(\alpha)}(x_j) - \varphi_j^{(\alpha)}(x_0)| + |\varphi_j^{(\alpha)}(x_0)| \longrightarrow 0,$$

because

$$|\varphi_j^{(\alpha)}(x_j) - \varphi_j^{(\alpha)}(x_0)| \le C|x_j - x_0| \to 0, \text{ and } \partial^\alpha \varphi_j(x_0) \to 0,$$

according to Lemmas 16.2 and 16.3. □

16.6. REMARK. Actually, we have proved a bit more than has been stated in Theorem 16.1. Namely, condition 2° follows from the proposition: $\langle f, \varphi_j \rangle \to 0$ as $j \to \infty$ for any $f \in L^1_{loc}(\Omega)$ and for any Sobolev derivative $f = \partial^\alpha g$, where $g \in L^1(\Omega)$.

Now we can define the spaces $\mathcal{D}$ and $\mathcal{D}'$ introduced by L. Schwartz [**54**].

16.7. DEFINITION. The space $\mathcal{D}(\Omega)$, which is sometimes called the *space of test functions* (compare with Section 1) is the space $C_0^\infty(\Omega)$ in which the following convergence of sequence of functions $\varphi_j \in C_0^\infty(\Omega)$ to a function $\varphi \in C_0^\infty(\Omega)$ is introduced:

a) there exists a compact K such that $\operatorname{supp} \varphi_j \subset K \ \forall j$;
b) $\forall \beta = (\beta_1, \dots, \beta_n) \ \forall \sigma > 0 \ \exists \ N = N(\beta, \sigma) \in \mathbb{N}$ such that

$$|\partial^\beta \varphi_j(x) - \partial^\beta \varphi(x)| < \sigma \quad \forall x \in \Omega \quad \text{for} \quad j \ge N.$$

In this case we write $\varphi_j \to \varphi$ in $\mathcal{D}$ as $j \to \infty$ (or $\lim_{j\to\infty} \varphi_j = \varphi$ in $\mathcal{D}$).

16.8. REMARK. It is clear that $\mathcal{D}(\Omega) = \bigcap\limits_{s \ge 0} \mathcal{D}_s(\Omega)$, where $\mathcal{D}_s(\Omega)$ is the space of functions $C_0^s(\Omega)$ equipped with a convergence which differs from the one introduced in Definition 16.7 only by the fact that the muliindex in condition *b*) satisfies the condition $|\beta| \le s$. One can show (see P.16.23) that $\mathcal{D}^\flat(\Omega) = \bigcup\limits_{s \ge 0} \mathcal{D}'_s(\Omega)$ (i.e., $f \in \mathcal{D}^\flat \iff \exists s \ge 0$ such that $f \in \mathcal{D}'_s$), where $\mathcal{D}'_s(\Omega)$ is the space of linear functional on $\mathcal{D}_s(\Omega)$ continuous with respect to this convergence in $\mathcal{D}_s(\Omega)$. The spaces $\mathcal{D}_s$ and $\mathcal{D}'_s$ have been introduced by Sobolev [**61**].

16.9. DEFINITION. The *space* $\mathcal{D}'(\Omega)$ *of the Schwartz distributions* (called also the *space of generalized functions*) is the space of linear continuous functional on $\mathcal{D}(\Omega)$, i.e., of linear functionals on $\mathcal{D}(\Omega)$ which are continuous in the convergence introduced in $\mathcal{D}(\Omega)$.

16.10. PROPOSITION. *There exists a sequence* $\{\epsilon_N\}_{N=1}^\infty$, $0 < \epsilon_N < 1$, *such that* $\lim\limits_{N\to\infty} r_N = 0$ *in* $\mathcal{D}$, *where* r_N *is defined in* (15.4).

PROOF. By the Teylor formula,

$$r_N(x) = a\left(\frac{x}{\epsilon_N}\right) \sum_{|\alpha|=N+1} \frac{N+1}{\alpha!} x^\alpha \int_0^1 (1-t)^{(N)} \varphi^{(\alpha)}(tx)dt.$$

This and the Leibniz formula imply that $|\partial^\beta r_N(x)| \le C_N(\epsilon_N)^{N-|\beta|} \le (1/2)^{N/2}$ for $N \ge N_0 = 2|\beta|$, if $\epsilon_N \le \frac{1}{2} C_N^{-2/N}$. $\square$

Propositions 15.3 and 16.10 imply

16.11. THEOREM (L. Schwartz). *If $f \in \mathcal{D}'(\Omega)$, $\operatorname{supp} f = 0 \in \Omega$, then there exist $N \in \mathbb{N}$ and $c_\alpha \in \mathbb{C}$ such that $f = \sum\limits_{|\alpha| \le N} c_\alpha \delta^{(\alpha)}$.*

16.12.**P.** *Let $f_k \in \mathcal{D}'(\mathbb{R})$, where $k = 0$ or $k = 1$, and $x \cdot f_k(x) = k$. Show (compare with Example 14.1) that $f_0(x) = C\delta(x)$, $f_1(x) = v.p.\frac{1}{x} + C\delta(x)$, where $C \in \mathbb{C}$.*

The following series of exercises P.16.13–P.16.25 concerns the question on the structure (general form) of distributions. Some hints are given at the end of the section.

16.13.**P.** *Verify that the following assertions are equivalent:*

a) *f is a distribution with a compact support, i.e., $f \in \mathcal{D}'(\Omega)$ and $\operatorname{supp} f$ is a compact in Ω;*
b) *$f \in \mathcal{E}'(\Omega)$, i.e., f is a linear continuous functional on $\mathcal{E}(\Omega)$, i.e., on the space $C^\infty(\Omega)$ with the following convergence: $\lim\limits_{j\to\infty} \varphi_j = \varphi$ in $\mathcal{E} \iff \lim\limits_{j\to\infty} a\varphi_j = a\varphi$ in $\mathcal{D}$ $\forall a \in C_0^\infty(\Omega)$.*

16.14. **P.** *Prove that $f \in \mathcal{D}'(\Omega)$ if and only if $f \in \mathcal{D}^\#(\Omega)$ and for any compact $K \subset \Omega$ there exist constants $C = C(K, f) > 0$ and $N = N(K, f) \in \mathbb{N}$ such that*

$$|\langle f, \varphi\rangle| \le C \cdot p_{K,N}(\varphi) \quad \forall \varphi \in C_0^\infty(K, \Omega) = \{\psi \in C_0^\infty(\Omega) \mid \operatorname{supp}\psi \subset K\}, \tag{16.5}$$

where

$$p_{K,N}(\varphi) = \sum_{|\alpha| \le N} \sup_{x \in K} |\partial^\alpha \varphi(x)|. \tag{16.6}$$

16.15. **P.** *(Compare with P.16.14). Let $\cup_{N \ge 1} K_N = \Omega$, where K_N are compacts in $\mathbb{R}^n$. Show that $f \in \mathcal{E}'(\Omega)$ (see P.16.13) if and* *only if* $f \in \mathcal{D}^{\#}(\Omega)$ *and there exist constants* $C = C(f) > 0$ *and* $N = N(f) \geq 1$ *such that* $|\langle f, \varphi\rangle| \leq C \cdot p_N(\varphi)$ $\forall \varphi \in C_0^\infty(\Omega)$, *where*

$$p_N(\varphi) = \sum_{|\alpha|\leq N} \sup_{x\in K_N} |\partial^\alpha \varphi(x)|. \tag{16.7}$$

16.16.**P.** *(Continuation). Let* $f \in \mathcal{E}'(\Omega)$, $\operatorname{supp} f \subset \omega \Subset \Omega \subset \mathbb{R}^n$. *Using* (16.7) *and noting that* $|\psi(x)| \leq \int_\Omega \left|\frac{\partial^n}{\partial x_1 \ldots \partial x_n}\psi(x)\right| dx$ $\forall \psi \in C_0^\infty(\Omega)$, *show that there exist numbers* $C > 0$ *and* $k \geq 1$ *such that*

$$|\langle f, \varphi\rangle| \leq C \int\limits_\Omega \left|\frac{\partial^{nm}}{\partial x_1^m \ldots \partial x_n^m}\varphi(x)\right| dx \quad \forall \varphi \in C_0^\infty(\omega). \tag{16.8}$$

16.17.**P.** *(Continuation). Checking that the function* $\varphi \in C_0^\infty(\omega)$ *can be uniquely recovered from its derivative* $\psi = \frac{\partial^{nm}}{\partial x_1^m \ldots \partial x_n^m}\varphi$, *show that the linear functional* $l : \psi \mapsto \langle f, \varphi\rangle$ *defined on the subspace* $Y = \{\psi \in C_0(\omega) \mid \psi = \frac{\partial^{nm}}{\partial x_1^m \ldots \partial x_n^m}\varphi,\ \varphi \in C_0^\infty)\}$ *of the space* $L^1(\omega)$ *is continuous.*

16.18.**P.** *(Continuation). Applying the Hahn–Banach theorem on continuation of linear continuous functionals (see, for instance,* [**36**]*), show that there exists a function* $g \in L^\infty(\omega)$ *such that*

$$\int\limits_\omega g(x) \frac{\partial^{nm}}{\partial x_1^m \ldots \partial x_n^m}\varphi(x)dx = \langle f, \varphi\rangle \quad \forall \varphi \in C_0^\infty(\omega).$$

16.19.**P.** *(Continuation). Show that the following theorems hold.*

16.20. THEOREM. (on the general form of distributions with a compact support). *Let* $f \in \mathcal{E}'(\Omega)$. *Then there exist a function* $F \in C(\Omega)$ *and a number* $M \geq 0$ *such that* $f = \partial^\alpha F$, *where* $\alpha = (M, \ldots, M)$, *i.e.,*

$$\langle f, \varphi\rangle = (-1)^{|\alpha|} \int\limits_\Omega F(x)\partial^\alpha \varphi(x)dx \quad \forall \varphi \in C_0^\infty(\Omega).$$

16.21. THEOREM (on the general form of distributions). *Let* $f \in \mathcal{D}'(\Omega)$. *Then there exists a sequence of functions* $F_\alpha \in C(\Omega)$, *parametrized by multiindices* $\alpha \in \mathbb{Z}_+^n$, *such that* $f = \sum\limits_\alpha \partial^\alpha F_\alpha$. *More exactly,* $F_\alpha = \sum\limits_{j=1}^\infty F_{\alpha_j}$, $F_{\alpha_j} \in C(\Omega)$, *and*

(1) $\operatorname{supp} F_{\alpha_j} \subset \Omega_j$, *where* $\{\Omega_j\}_{j\geq 1}$ *is a locally finite cover of* Ω*;*

(2) $\forall j \geq 1\ \exists M_j \geq 1$ *such that* $F_{\alpha_j} = 0$ *for* $|\alpha| > M_j$.

16.22.**P.** *(Peetre [**47**]). Let* $A : \mathcal{E}(\Omega) \to \mathcal{E}(\Omega)$ *be a linear continuous operator with the localization property, i.e.,* $\operatorname{supp}(Au) \subset \operatorname{supp}(u)$ $\forall u \in \mathcal{E}(\Omega)$. *Then* A *is a differential operator, more exactly: there exists a family* $\{a_\alpha\}_{\alpha\in\mathbb{Z}_+^n}$ *of functions* $a_\alpha \in C^\infty(\Omega)$ *such that, for* $u \in \mathcal{E}(\Omega)$, $(Au)(x) = \sum\limits_{|\alpha|\leq m(x)} a_\alpha(x)\partial^\alpha u(x)$, *where* $m(x) \leq N(K) < \infty$ *for any compact* $K \subset \Omega$.

16.23.**P.** *(See Remark 16.8). Check that* $\mathcal{D}^\flat(\Omega) = \bigcup\limits_s \mathcal{D}'_s(\Omega)$.

16.24. Remark. We say that a functional $f \in \mathcal{D}'$ has a finite order of singularity, if there exist $k \geq 1$ and functions $f_\alpha \in L^1_{loc}$, where $|\alpha| \leq k$, such that $f = \sum\limits_{|\alpha|=k} \partial^\alpha f_\alpha$. The least k, for which such a representation of f is possible, is called its order of singularity. In these terms, the space of Sobolev derivatives $\mathcal{D}^\flat$ is, according to Definition 12.2, the space of all distributions which have a finite order of singularity.

16.25.**P.** *Resolve the following paradox. On one hand, the discontinuous function*

$$f(x,y) = \begin{cases} \Re\exp(-1/z^4) & \text{for } z \neq 0, \quad z = x+iy \in \mathbb{C}, \\ 0 & \text{for } z = 0 \end{cases} \tag{16.9}$$

(being the real part of a function analytic in $\mathbb{C}\setminus 0$ *with zero second derivative with respect to* x *and* y *at the origin) is a solution of the Laplace equation on the plane. On the other hand, by virtue of Theorem 16.20 and a priori estimate* (21.6) *(see also Corollary 22.24 and* [**22**, no 2, § 3, item 6]*): if* $f \in \mathcal{D}'(\Omega)$ *and* $\Delta f \equiv 0$ *in* Ω, *then* $f \in C^\infty(\Omega)$.

16.26.**P.** *Show that* $\mathcal{E}$ *is metrizable and* $\mathcal{D}$ *is non-metrizable.*

16.27. Remark. One can introduce in $\mathcal{D}$ (respectively, in $\mathcal{E}$) a structure of so-called (see [**36, 51**]) linear locally convex topological space (LLCPS[3]) and such that the convergence in this space coincides with the one introduced above. For instance, the neighbourhood of zero in $\mathcal{D}$ can be defined with the help of any finite set of everywhere positive functions $\gamma_m \in C(\Omega)$ $(m = 0, 1, \dots, M; M \in \mathbb{Z}_+)$ as the set of all the functions $\varphi \in C_0^\infty(\Omega)$ such that $|\partial^\alpha \varphi| < \gamma_{|\alpha|}$, if $|\alpha| \le M$. The topology in $\mathcal{E}$ can be defined by introducing the distance by the formulae given in the hint to P.16.26. Thus, $\mathcal{E}$ is the *Frechét space*, i.e., a complete metric LLCTS. One can extend to the Frechét spaces (see, for instance, [**51**]) the Banach–Steinhaus theorem: a space of linear continuous functionals on a Frechét space (in particular, the space $\mathcal{E}'$) is complete relatively the weak convergence. Although $\mathcal{D}$ is not a Frechét space (by virtue of P.16.26), $\mathcal{D}'$ is also complete relatively the weak convergence (the direct proof see, for instance, in [**22**]).

[3)] A linear space X is called a *linear locally convex topological space*, if this space is *topological* [**36**], the operations of addition and multiplication by a number are continuous and, moreover, any neighbourhood of zero in X contains a CONVEX neighbourhood of zero.

Hints to P.16.13–P.16.26:

16.13. If we suppose that b) does not imply a), then there exists a sequence of points x_k such that $x_k \to \partial\Omega$, and $f \ne 0$ in the vicinity of x_k.

16.14. If $f \in \mathcal{D}'$ but estimate (16.5) is not valid, then $\exists K = \bar{K} \subset \Omega$ $\forall N \ge 1$ $\exists \varphi \in C_0^\infty(\Omega)$, $\operatorname{supp}\varphi \subset K_N$, and $|\langle f, \varphi_N\rangle| \ge N \sum\limits_{|\alpha|\le N} \sup\limits_K |\varphi_N^{(\alpha)}|$. We have $\psi_N = \varphi_N \cdot |\langle f, \varphi_N\rangle|^{-1} \to 0$ in $\mathcal{D}$ but $|\langle f, \psi_N\rangle| = 1$.

16.15. Since $\langle f, \varphi\rangle = \langle f, \rho\varphi\rangle$, where $\rho \in C_0^\infty$, $\rho \equiv 1$ on $\operatorname{supp} f$, it follows that $K = \operatorname{supp}\rho$. Warning: in general case, $K \ne \operatorname{supp} f$. Indeed, following [**54**, vol. 1, p. 94], consider the functional $f \in \mathcal{E}'(\mathbb{R})$ defined by the formula

$$\langle f, \varphi\rangle = \lim_{m\to\infty} \left[\left(\sum_{\nu \le m} \varphi(1/\nu) \right) - m\varphi(0) - (\ln m)\varphi'(0) \right].$$

Obviously, $\operatorname{supp} f$ is the set of the points of the form $1/\nu$, $\nu \ge 1$, together with their limit point $x = 0$. Consider the sequence of functions $\varphi_j \in C_0^\infty(\mathbb{R})$ such that $\varphi_j(x) = 0$ for $x \le \frac{1}{j+1}$, $\varphi_j(x) = 1/\sqrt{j}$ for $1/j \le x \le 1$. Taking $K = \operatorname{supp} f$ in (16.6), we have $p_{K,N}(\varphi_j) \to 0$ as $j \to \infty$ $\forall N \ge 1$, while $\langle f, \varphi\rangle = j/\sqrt{j} \to \infty$.

16.16. $\sup_K |\partial^\alpha\varphi(x)| \le C_j(K) \sup_K \left|\frac{\partial}{\partial x}\partial^\alpha\varphi(x)\right|$.

16.17. Apply (16.8).

16.18. By the Riesz theorem (see Theorem 9.14), $(L^1)' = L^\infty$.

16.20. Complete the definition of g outside ω by zero (see P.16.18) and take $F(x) = (-1)^{mn}\int_{y<x} g(y)\,dy$.

16.21. Let $\sum\psi_j \equiv 1$ be a partition of unity. We have

$$\langle f,\varphi\rangle = \sum_j \langle\psi_j f,\varphi\rangle = \sum_j \sum_{|\alpha|\le M_j} \langle\partial^\alpha F_{\alpha_j},\varphi\rangle = \sum_\alpha \langle\partial^\alpha \sum_j F_{\alpha_j},\varphi\rangle.$$

16.22. Verify that the functional $A_a : \mathcal{E} \ni u \mapsto Au\big|_{x=a}$ belongs to $\mathcal{E}'$ and $\operatorname{supp} A_a = a$. Thus, $Au\big|_{x=a} = \sum\limits_{|\alpha|\le m(a)} (a_\alpha(x)\partial^\alpha u)\big|_{x=a}$ Using the Banach–Steinhaus theorem (see note 1 in Section 16), prove that $\sup\limits_{a\in K} m(a) < \infty$ for any $K \subset \bar{K} \subset \Omega$. Applying A to $(y-x)^\alpha/\alpha!$, show that $a \in C^\infty(\Omega)$.

16.23. Apply Theorem 16.21.

16.25. Function (16.9) does not belong to $\mathcal{D}'$ (i.e., does not admit the regularization in $\mathcal{D}'$) as well as any other function $f \in C^\infty(\mathbb{R}\setminus 0)$ which for any $m \in \mathbb{N}$ and $C > 0$ does not satisfies the estimate $|f(x)| \le C|x|^{-m}$ for $0 < |x| < \epsilon$, where $1/\epsilon \gg 1$. The last fact can be proved, by constructing a sequence of numbers $\epsilon_j > 0$ such that , for the function $\varphi_j(x) = \epsilon_j\varphi(jx)$, where $\varphi \in C_0(\mathbb{R})$, $\varphi = 0$ outside the domain $\{1 < |x| < 4\}$, $\int\varphi = 1$, the following conditions are satisfied: $\int_{\mathbb{R}^n} f(x)\varphi_j(x)dx \to \infty$ as $j \to \infty$ but $\varphi_j \to 0$ in $\mathcal{D}$ as $j \to \infty$.

16.26. The distance in $\mathcal{E}$ can be given by the formula $\rho(\varphi,\psi) = d(\varphi-\psi)$, where $d(\varphi) = \sum\limits_1^\infty 2^{-N}\min(p_N(\varphi),1)$, and p_N is defined in (16.7). $\mathcal{D}$ is non-merizable since for the sequence $\varphi_{k,m}(x) = \varphi(x/m)/k$, where $\varphi \in \mathcal{D}(\mathbb{R})$, the following property, which is valid in any metric space, does not hold: if $\varphi_{k,m} \to 0$ as $k \to \infty$, then $\forall m$ $\exists k(m)$ such that $\varphi_{k(m),m} \to 0$ as $m \to \infty$.

CHAPTER 3

The spaces H^s. Pseudodifferential operators

17. The Fourier series and the Fourier transform. The spaces $\mathcal{S}$ and $\mathcal{S}'$

In 1807, Jean Fourier said his word in the famous discussion (going from the beginning of 18th century) on the sounding string [**50**]. Luzin wrote [**42**] that he accomplished a discovery which "made a great perplexity and confusion among all the mathematicians. It turned over all the notions" and became a source of new deep ideas for development of the concepts of a function, an integral, a trigonometric series and so on. Fourier's discovery (however strange it seems at first sight) consists in the formal rule of calculations of the coefficients

$$a_k = \frac{1}{p} \int\limits_{-p/2}^{p/2} u(y) e^{-\mathring{\imath}(k/p)y} dy, \quad \mathring{\imath} = 2\pi i, \ i = \sqrt{-1} \tag{17.1}$$

(which are called the Fourier coefficients) in the "expansion"

$$u(x) \sim \sum_{k=-\infty}^{\infty} a_k e^{\mathring{\imath}(k/p)x}, \quad |x| < p/2 \tag{17.2}$$

of an "arbitrary" function $u : \Omega \ni x \longmapsto u(x) \in \mathbb{C}$, where $\Omega =]-p/2, p/2[$, by the harmonics

$$e^{\mathring{\imath}(k/p)x} = \cos 2\pi(k/p)x + i \sin 2\pi(k/p)x, \quad k \in \mathbb{Z}. \tag{17.3}$$

The trigonometric series (17.2) is called the *Fourier series* of the function u (more exactly, the Fourier series of the function u in the system of functions (17.3)$^{1)}$). The first result concerning the convergence of the Fourier series was obtained by 24 years old L. Dirichlet in 1829 (see, for instance, [**72**]): if u is piecewise continuous on $[-p/2, p/2]$ and the number of its intervals of monotonicity is finite, then the Fourier series of the function u converges to u at every point of continuity of u; moreover, if the function f is continuous and $u(-p/2) = u(p/2)$, then series (17.2) converges to u uniformly. Although the Fourier coefficients (17.1) are defined for any function $u \in L^1$, the Fourier series can diverge at some points even for continuous functions (see, for instance, [**36, 56**]; also compare with P.17.9). As for integrable functions, in 1922, 19 years old A.N. Kolmogorov constructed [**37**] the famous example of a function $u \in L^1$, whose Fourier series diverges almost everywhere (and later an example of the Fourier series, everywhere diverging, of an integrable function).

[1)] See in this connection formulae (17.17)–(17.18).

The following theorem (see, for instance, [**56**]) on convergence of the Fourier series in the space L^2 is of great importance: "for any $u \in L^2(\Omega)$, where $\Omega =]-p/2, p/2[$, series (17.2) converges to u in $L^2(\Omega)$", i.e.,

$$\|u - \sum_{|k| \le N} a_k e_k\|_{L_2} \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty, \tag{17.4}$$

where

$$e_k : \Omega \ni x \longmapsto e_k(x) = \exp\left(\mathring{i}\frac{k}{p}x\right). \tag{17.5}$$

This theorem shows the transparent geometric meaning of the Fourier coefficients. Actually, consider in $L^2(\Omega) \times L^2(\Omega)$ the complex-valued functional

$$(u, v) = \int_{-p/2}^{p/2} u(x)\bar{v}(x)dx,$$

where $\bar{v}$ is the complex-conjugate function to v. Obviously, this functional defines a *scalar product*[2)] in the space $L^2(\Omega)$ with respect to which (one can readily see) functions (17.5) are orthogonal, i.e.,

$$(e_k, e_m) = 0 \quad \text{for} \quad k \ne m. \tag{17.6}$$

Therefore, by choosing $N \ge |m|$ and multiplying scalarly the function $(u - \sum_{|k| \le N} a_k e_k)$ by e_m, we obtain $|(u, e_m) - a_m \cdot (e_m, e_m)| =$

$|(u - \sum_{|k| \le N} a_k e_k, e_m)| \le \|u - \sum_{|k| \le N} a_k e_k\|_{L^2} \|e_m\|_{L^2}$. This and (17.4) imply that

$$a_m = (u, e_m)/(e_m, e_m), \quad m \in \mathbb{Z}. \tag{17.7}$$

Thus, the coefficient a_k is the algebraic value of the orthogonal projection of the vector $u \in L^2(\Omega)$ onto the direction of the vector e_k.

2) This means that the functional (u, v) is linear in the first argument and $(u, u) > 0$, if $u \neq 0$, and $(u, v) = \overline{(v, u)}$, where the bar denotes the complex conjugation. Note that the function $u \longmapsto \|u\| = \sqrt{(u, u)}$ is a norm (see note 5 in Section 8), and $|(u, v)| \le \|u\| \cdot \|v\|$ (compare with (9.3) for $p = 2$). Recall also that the Banach space X (see note 5 in Section 8) with a norm $\|\cdot\|$ is called a *Hilbert space*, if in X there exists a scalar product $(\cdot, \cdot)$ such that $(x, x) = \|x\|^2$ $\forall x \in X$. Thus, $L^2(\Omega)$ is a Hilbert space.

When the geometric meaning of the Fourier coefficients became clear, it might seem surprising that, as Luzin wrote, "neither subtle analytical intellect of d'Alembert nor creative efforts of Euler, D. Bernoulli and Lagrange can solve this most difficult problem[3)], i.e., the problem concerning the formulae for the coefficients a_k in (17.2). However, one should not forget that the geometric transparency of formulae (17.7) given above became possible only thanks to the fact that the Fourier formulae (17.1) put on the agenda the problems whose solution allowed to give an exact meaning to the words such as "function", "representation of a function by a trigonometric series" and many, many others.

3) The reason of arising this question is historically connected with the problem of a sounding string (see [**42, 50**]) — the first system with an infinite number of degrees of freedom which was mathematically investigated. As far back as in 1753, D. Bernoulli came to the conclusion that the most general motion of a string can be obtained by summing the principal oscillations. In other words, the general solution $u = u(x, t)$ of the differential equation of a string

$$u_{tt} - u_{xx} = 0, \quad |x| < p/2, \ t > 0 \tag{17.8}$$

which satisfies, for instance, the periodicity condition

$$u(-p/2, t) - u(p/2, t) = 0, \ u_x(-p/2, t) - u_x(p/2, t) = 0 \tag{17.9}$$

can be represented in the form of a sum of harmonics propagating to the right and to the left (along the characteristics $x \pm t = 0$, compare with Section 11), more exactly:

$$u(x,t) = \sum_{k \in \mathbb{Z}} [a_k^+ e^{i\lambda_k(x+t)} + a_k^- e^{i\lambda_k(x-t)}], \tag{17.10}$$

where $a_k^\pm \in \mathbb{C}$, and $\lambda_k = 2\pi k/p$. Indeed, equation (17.8) and the boundary conditions (17.9) are linear and homogeneous. Therefore, a linear combination of functions satisfying (17.8)–(17.9) satisfies these conditions as well. This fact suggests an idea to find the general solution of problem (17.8)–(17.9), by summing (with indeterminate coefficients) the particular solutions of equation (17.8), which satisfy the periodicity conditions (17.9). Equation (17.8) belongs to those which have an infinite series of particular solutions with *separated variables*, i.e., non-zero solutions of the form $\varphi(x)\psi(t)$. Actually, substituting this function into (17.8), we obtain $\varphi_{xx}(x)\psi(t) = \varphi(x)\psi_{tt}(t)$. Hence,

$$\varphi_{xx}(x)/\varphi(x) = \psi_{tt}(t)/\psi(t) = \text{const}\,. \tag{17.11}$$

The periodicity condition (17.9) implies that $\varphi \in X$, where

$$X = \{\varphi \in C^2(\Omega) \cap C^1(\bar{\Omega}) \mid \varphi(-\tfrac{p}{2},t) = \varphi(\tfrac{p}{2},t),\ \varphi'(-\tfrac{p}{2},t) = \varphi'(\tfrac{p}{2},t)\}. \tag{17.12}$$

Thus, the function φ necessarily (see (17.11)) must be an *eigenfunction* of the operator

$$-\,d^2/dx^2 : X \longrightarrow L^2(\Omega), \quad \Omega =]-p/2, p/2[. \tag{17.13}$$

This means that φ is a non-zero function of the space X, satisfying the condition

$$-\,d^2\varphi/dx^2 = \mu \cdot \varphi, \tag{17.14}$$

for a constant $\mu \in \mathbb{C}$, that is called the *eigenvalue* of operator (17.13). Since φ belongs to the space X, it follows that the number μ can be only positive (because otherwise $\varphi \equiv 0$). Let us denote μ by λ^2. Then (17.14) implies that $\varphi(x) = ae^{i\lambda x}$, $\lambda \in \mathbb{R}$, $a \in \mathbb{C} \setminus \{0\}$. Obviously, this formula is consistent with the condition $\varphi \in X$ if and only if $\lambda = \lambda_k = 2k\pi/p$, $k \in \mathbb{Z}$. Thus, taking into account (17.11), we obtain

$$\varphi_k(x)\psi_k(t) = a_k^+ e^{i\lambda_k x} e^{i\lambda_k t} + a_k^- e^{i\lambda_k x} e^{-i\lambda_k t},\ \alpha_k^\pm \in \mathbb{C},$$

and, hence, D. Bernoulli's formula (17.10) is established.

The Bernoulli formula brought into use the principle of composition of oscillations as well as put many serious mathematical problems. One of them is connected with the finding of the coefficients $a_k^\pm$ in formula (17.10) for any specific oscillation that (compare with P.11.21) is determined by the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \tag{17.15}$$

i.e., by the initial deviation of the string from the equilibrium position and by the initial velocity of the motion of its points. In other words, the Bernoulli formula posed the question of finding the coefficients $a_k^{\pm}$ from the condition

$$\sum_{k\in\mathbb{Z}}(a_k^+ + a_k^-)e^{i\lambda_k x} = f(x), \ \sum_{k\in\mathbb{Z}} i\lambda_k(a_k^+ - a_k^-)e^{i\lambda_k x} = g(x).$$

It is interesting that in 1759, i.e., in six years after the work by D. Bernoulli, formulae (17.1) answering the formulated question were almost found by 23-year Lagrange. It remained to him only to make in his investigations the rearrangement of the limits in order to obtain these formulae. However, as Luzin writes, “Lagrange's thought was directed in another way and he, almost touching the discovery, so little realized it that he flung about D. Bernoulli the remark “It is disappointing that such a witty theory is inconsistent.”

As has been said, half a century later the answer to this question was given by Fourier who wrote formulae (17.1). This is the reason why the method, whose scheme was presented on the example of solution of problem (17.8)–(17.9), (17.15), is called the *Fourier method* (see, for instance, [**25, 68**]). The term is also used (by obvious reasons) the *method of separation of variables.* This method is rather widespread in mathematical physics.

The reader can easily find by the Fourier method the solution of the Dirichlet problem for the Laplace equation in a rectangle, by considering preliminary the special case:

$$\Delta u(x,y) = 0, \quad (x,y) \in]0,1[^2; \quad u\big|_{x=0} = u\big|_{x=1} = 0;$$
$$u\big|_{y=0} = f(x), \quad u\big|_{y=1} = g(x).$$

Equally easy one can obtain by the Fourier method the solution

$$u(x,t) = u_N(x,t) + \sum_{k>N}^{\infty} \frac{2\sin\lambda_k}{\lambda_k[1+\sigma\sin^2\lambda_k]} e^{-\lambda_k^2 t}\cos\lambda_k x, \ N \geq 0, \ u_0 \equiv 0 \tag{17.16}$$

of problem (6.11) for the heat equation. In formula (17.16), λ_k^2 is the kth ($k \in \mathbb{N}$) eigenvalue of the operator

$$-d^2/dx^2 : Y \longrightarrow L^2(\Omega), \quad \Omega =]0,1[,$$

defined on the space

$$Y = \{\varphi \in C^2(\Omega) \cap C^1(\bar{\Omega}) \mid (\varphi \pm \sigma\varphi')\big|_{x=\pm 1} = 0\},$$

where $\sigma \geq 0$ is the parameter of problem (6.11). Note that $\lambda_k \in](k-1)\pi, (k-1/2)\pi]$ is the kth root of the equation $\operatorname{cotan}\lambda = \sigma\lambda$ and the eigenfunctions $\varphi_k(x) = \cos\lambda_k x$ of operator (17.16) satisfy (compare with (17.6)) the orthogonality condition:

$$(\varphi_k, \varphi_m) = \int_{-1}^{1} \varphi_k(x)\varphi_m(x)dx = 0 \quad \text{for} \quad k \neq m. \tag{17.17}$$

Indeed, integrating by parts (or applying the Ostrogradsky–Gauss formula in the multidimensional case) and taking into account the boundary conditions $(\varphi \pm \sigma\varphi')\big|_{x=\pm 1} = 0$ and the fact that $-\varphi_k'' = \lambda_k^2\varphi_k$, we have

$$\begin{aligned}(\lambda_k^2 - \lambda_m^2)(\varphi_k, \varphi_m) &= \int_{-1}^{1} (\varphi_k\varphi_m'' - \varphi_m\varphi_k'')\,dx \\ &= \varphi_k\varphi_m'\big|_{-1}^{1} - \int_{-1}^{1} \varphi_k'\varphi_m' - \varphi_m\varphi_k'\big|_{-1}^{1} + \int_{-1}^{1} \varphi_k'\varphi_m' = 0.\end{aligned}$$

One can show (see, for instance, [**68**]) that the eigenfunctions φ_k, $k \in \mathbb{N}$, form (compare with (17.4)) a *complete system* in $L^2 = \bar{Y}$, i.e., $\forall u \in L^2$ $\forall\epsilon > 0$ there exist $N \geq 1$ and numbers $a_1, \ldots, a_N$ such that $\|u - \sum\limits_{k=1}^{N} a_k\varphi_k\|_{L^2} < \epsilon$. Therefore, (compare with (17.2)–(17.7)) the formal series

$$\sum_{k=1}^{\infty} \frac{(u, \varphi_k)}{(\varphi_k, \varphi_k)}\varphi_k, \tag{17.18}$$

which is called by the *Fourier series* of the function u in the *orthogonal* (by virtue of (17.17)) *system of functions* φ_k, converges to u in L^2. The reader can easily verify that $(1, \varphi_k)/(\varphi_k, \varphi_k) = 2\sin\lambda_k/(\lambda_k[1 + \sigma\sin^2\lambda_k])$ as well as that series (17.16) converges uniformly together with all its derivatives for $t \geq \epsilon$ for any $\epsilon > 0$ and determines a smooth (except the angle points $(x, t) = (\pm 1, 0)$) and unique (see, for instance, [**20**]) solution of problem (6.11).

Let us note one more circumstance. Series (17.16) converges quickly for large t. One can show that, for any $k \geq 1$,

$$|u(x,t) - u_N(x,t)| < 10^{-k}/N \quad \textit{for } t > k/(4.3N^2). \tag{17.19}$$

However, for small t, series (17.16) converges very slowly. Therefore, for small t, it is advisable to use another representation of the solution of problem (6.11) that is obtained below in Section 18 with the help of concept of dimensionality (see Section 6) and the so-called Laplace transform.

Substituting formally (17.1) into (17.2), we obtain

$$u(x) = \sum_{k=-\infty}^{\infty} \frac{1}{p} e^{\mathring{i}(k/p)x} \int_{-p/2}^{p/2} e^{-\mathring{i}(k/p)y} u(y)dy. \tag{17.20}$$

Tending p to infinity and passing formally in (17.20) to the limit, we obtain, as a result, for the "arbitrary" function $u : \mathbb{R} \to \mathbb{C}$ the (formal) expression

$$u(x) = \int_{-\infty}^{\infty} e^{\mathring{i}x\xi} \Big(\int_{-\infty}^{\infty} e^{-\mathring{i}y\xi} u(y)dy \Big) d\xi. \tag{17.21}$$

From these formal calculations we may give the exact

17.1. Definition. Let $\xi \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, $x\xi = \sum_{k=1}^{n} x_k\xi_k$, i.e., $x\xi = (x, \xi)$ is the scalar product of x and ξ. The function

$$\widetilde{u}(\xi) = \int_{\mathbb{R}^n} e^{-\mathring{i}x\xi} u(x)dx, \ \ \mathring{i} = 2\pi i, \ \ i = \sqrt{-1} \tag{17.22}$$

is called the *Fourier transform* of the function $u \in L^1(\mathbb{R}^n)$, and the mapping $\mathbf{F} : L^1(\mathbb{R}^n) \ni u \longmapsto \widetilde{u} = \mathbf{F}u \in \mathbb{C}$ is called the *Fourier transformation* (in $L^1(\mathbb{R}^n)$).

17.2. Lemma. *If $u \in L^1(\mathbb{R}^n)$, then $\mathbf{F}u \in C(\mathbb{R}^n)$ and $\|\mathbf{F}u\|_C \leq \|u\|_{L_1}$.*

Proof. It follows from the (Lebesgue) theorem 8.20 that $\widetilde{u} = \mathbf{F}u \in C(\mathbb{R})$; moreover, $|\widetilde{u}(\xi)| \leq \int_{\mathbb{R}^n} |u(x)|\, dx$. $\square$

17.3. Example. Let $u_\pm(x) = \Theta_\pm(x)e^{\mp ax}$, where $x \in \mathbb{R}$, $a > 0$, and $\Theta_\pm$ is defined in (12.5). Then $\widetilde{u}_\pm(\xi) = \frac{1}{a \pm \mathring{i}\xi}$. Let us note that $\widetilde{u}_\pm \notin L^1$ although $u_\pm \in L^1$. Note as well that the function $\widetilde{u}_\pm$ can be analytically continued into the complex half-plane $\mathbb{C}_\mp$.

Below, in Theorem 17.6 we give some conditions under which expression (17.21) acquires the exact meaning of one of the most important formulae in analysis. Preliminary, we give

17.4. Definition. Let $p \geq 1$ and $k \in \mathbb{Z}$. We say that a function $u \in L^p(\Omega)$ belongs to the *Sobolev space* $W^{p,k}(\Omega)$, if all its Sobolev derivatives $\partial^\alpha u$, where $|\alpha| \leq k$, belong to $L^p(\Omega)$. The convergence in the space $W^{p,k}$ is characterized by the norm

$$\|u\|_{W^{p,k}} = \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^p}, \tag{17.23}$$

i.e., $u_j \to u$ in $W^{p,k}$ as $j \to \infty$, if $\|u - u_j\|_{W^{p,k}} \to 0$ as $j \to \infty$.

One can readily verify that $W^{p,k}$ is a Banach space.

17.5. Lemma. [4] *$W^{1,n}(\mathbb{R}^n) \subset C(\mathbb{R}^n)$, i.e., for any element $\{u\} \in W^{1,n}$ there exists a unique function $u \in C$ which coincides almost everywhere with any representative*[5] *of the element $\{u\}$, and $\|u\|_C \leq \|u\|_{W^{1,n}}$.*

[4] Lemma 17.5 is a simple special case of the Sobolev embedding theorem (see, for instance, [**8, 40, 62, 70**]). Note that the embedding $W^{p,k}(\mathbb{R}^n) \subset C(\mathbb{R}^n)$, valid for $n/p < k$, does not hold if $p > 1$ and $n/p = k$ (see, in particular, Section 20, where the special case $p = 2$ is considered).

[5] See note 1 in Section 9.

Proof. It follows from the (Fubini) theorem 8.24 and Theorem 8.27 that the function u admits the representation in the form

$$u(x) = \int_{-\infty}^{x_1} \left[\int_{-\infty}^{x_2} \dots \left[\int_{-\infty}^{x_n} \frac{\partial^n u(y_1, \dots y_n)}{\partial y_1 \partial y_2 \dots \partial y_n} dy_n \right] \dots dy_2 \right] dy_1,$$
$$x = (x_1, \dots x_n),$$

that implies its continuity and the estimate $\|u\|_C \leq \int |\frac{\partial^n u(x)\, dx}{\partial x_1 \dots \partial x_n}|$. $\square$

17.6. Theorem. *Let $u \in W^{1,n}(\mathbb{R}^n)$. Then, for any $x \in \mathbb{R}^n$,*

$$u(x) = \lim_{N \to \infty} u_N(x), \quad \textit{where} \quad u_N(x) = \int_{-N}^{N} \dots \int_{-N}^{N} e^{\mathring{i} x \xi} \widetilde{u}(\xi) d\xi_1 \dots d\xi_n, \tag{17.24}$$

and $\widetilde{u} = \mathbf{F} u$ is the Fourier transform of the (continuous[6]*) function u.*

[6] By virtue of Lemma 17.5.

PROOF. It follows from the Fubini theorem that

$$u_N(x) = \int_{-\infty}^{\infty} \left[\dots \left[\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} u(y) \frac{\partial \theta_N(y_1 - x_1)}{\partial y_1} dy_1 \right] \right. \right.$$
$$\left. \left. \times \frac{\partial \theta_N(y_2 - x_2)}{\partial y_2} dy_2 \right] \dots \right] \frac{\partial \theta_N(y_n - x_n)}{\partial y_n} dy_n,$$

where $\theta_N(\sigma) = \int_{-1}^{\sigma} \delta_N(s)ds$ and $\delta_N(s) = \int_{-N}^{N} e^{\mathring{\imath} s\xi} d\xi = \frac{\sin 2\pi N s}{\pi s}$. Note (compare with P.4.3 and P.13.11) that $\theta_N(\sigma) \to \theta(\sigma)$, $\sigma \in \mathbb{R}$, and $\forall N$ $|\theta_N(\sigma)| \le \text{const}$. Actually, setting $\lambda_k = \int_{k/2N}^{(k+1)/2N} \delta_N(\sigma)d\sigma$ for $k \in \mathbb{Z}_+$, we have that λ_k does not depend on N, $|\lambda_k| \downarrow 0$ as $k \to \infty$, moreover, $\lambda_{2k} > -\lambda_{2k+1}$ and $\sum\limits_{k=0}^{\infty} \lambda_k = \frac{1}{\pi} \int_0^{\infty} x^{-1} \sin x \, dx = \frac{1}{2}$. Therefore, $\theta_N(\sigma) \to \theta(\sigma)$ and $|\theta_N(\sigma)| \le 2\lambda_0$. Then (as in the proof of Lemma 13.10) we should integrate by parts, apply the Lebesgue theorem and obtain that $u_N(x) \to u(x)$. □

17.7. REMARK. The proof of the theorem containing, in particular, the solution of Exercise P.4.3 (and P.13.11) shows (taking into account the proof of Lemma 17.5) that the assertion of Theorem 17.6 is valid under more broad hypotheses: it is sufficient to require that the function u and n its derivatives $\partial^k u/\partial x_1 \partial x_2 \dots \partial x_k$, $k = 1, \dots, n$, be integrable in $\mathbb{R}^n$.

17.8. REMARK. Of course, the assertion of Theorem 17.6 makes the sense only for $u \in L^1 \cap C$. However, this necessary condition is not sufficient for validity of (17.24), as it follows from

17.9.**P.** *Constructing (compare with* [**36**]*) a sequence of functions* $\varphi_N \in L^1(\mathbb{R}) \cap C(\mathbb{R})$*, such that* $\int_{-\infty}^{\infty} y^{-1} \sin Ny \varphi_N(y) dy \to \infty$ *(as* $N \to \infty$*) and* $\|\varphi_N\|_{L_1} + \|\varphi_N\|_C \le 1$ *and applying the Banach–Steinhaus theorem (see note* 1 *in Section 16), show that there exists a function* $\varphi \in L^1(\mathbb{R}) \cap C(\mathbb{R})$*, for which equality* (17.24) *does not hold at least at one point* $x \in \mathbb{R}$*.*

Formal expression (17.21) and Theorem 17.6 suggest the idea on expediency of introducing the transformation

$$\mathbf{F}^{-1} : L^1(\mathbb{R}) \ni \widetilde{u} \longmapsto \mathbf{F}^{-1}\widetilde{u} \in \mathbb{C},$$

defined by the formula

$$(\mathbf{F}^{-1}\widetilde{u})(x) = \int_{\mathbb{R}^n} e^{\mathring{i}x\xi}\widetilde{u}(\xi)d\xi,\ \mathring{i} = 2\pi i,\ i = \sqrt{-1},\ x \in \mathbb{R}^n. \quad (17.25)$$

This formula differs from formula (17.22) by the sign of the exponent. The transformation $\mathbf{F}^{-1}$ is called the *inverse Fourier transform*, since $u = \mathbf{F}^{-1}\mathbf{F}u$, if $u \in W^{1,n}(\mathbb{R}^n)$ and $\mathbf{F}u \in L^1(\mathbb{R})$.

Define (following L. Schwartz [**54**]) the *space of rapidly decreasing functions* $\mathcal{S} = \mathcal{S}(\mathbb{R}^n) \subset W^{1,n}(\mathbb{R}^n)$. In the space $\mathcal{S}$ (see Theorem 17.16 below), the transformations $\mathbf{F}^{-1}$ and $\mathbf{F}$ are automorphisms (i.e., linear invertible mappings from $\mathcal{S}$ onto itself).

17.10. Definition. The elements of the space $\mathcal{S}(\mathbb{R}^n)$ are the functions $u \in C^\infty(\mathbb{R}^n)$ that satisfy the following condition: for any multiindices $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$, there exists a number $C_{\alpha\beta} < \infty$ such that $\forall x = (x_1, \dots, x_n) \in \mathbb{R}^n$

$$|x^\alpha \partial_x^\beta u(x)| \le C_{\alpha\beta}, \quad \text{where} \quad x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n},\ \partial_x^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}}.$$

In this case, we say that the sequence of functions $u_j \in \mathcal{S}$ converges in $\mathcal{S}$ to u $(u_j \to u$ in $\mathcal{S})$ as $j \to \infty$, if $\forall \epsilon > 0$ $\forall m \in \mathbb{N}$ $\exists j_0 \in \mathbb{N}$ $\forall j \ge j_0$ the following inequality holds: $p_m(u_j - u) \le \epsilon$, where

$$p_m(v) = \sup_{x \in \mathbb{R}^n} \left(\sum_{|\alpha| \le m} (1 + |x|)^m |\partial^\alpha v(x)| \right)$$

Obviously, $e^{-x^2} \in \mathcal{S}(\mathbb{R})$ but $e^{-x^2}\sin(e^{x^2}) \notin \mathcal{S}(\mathbb{R})$.

17.11. **P.** *Check that the space $\mathcal{S}$ is a Frechét space (see Remark 16.27) in which the distance ρ can be defined by the formula:*

$$\rho(u, v) = d(u - v), \quad \textit{where} \quad d(\varphi) = \sum_{m=1}^{\infty} 2^{-m} \inf(1, p_m(\varphi)).$$

17.12. **P.** *Show (see P.16.13) that $\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset \mathcal{E}(\mathbb{R}^n)$. In particular, show that the convergence in $\mathcal{D}$ (in $\mathcal{S}$) implies the convergence in $\mathcal{S}$ (in $\mathcal{E}$). Verify that $\mathcal{D}$ is dense in $\mathcal{S}$ and $\mathcal{S}$ is dense in $\mathcal{E}$.*

17.13. **P.** *Integrating by parts, verify that the following lemma holds.*

17.14. Lemma. *For any multiindices α, β and any $u \in \mathcal{S}$*

$$(-\mathring{i})^{|\beta|}\mathbf{F}[\partial_x^\alpha(x^\beta u(x))](\xi) = (\mathring{i})^{|\alpha|}\xi^\alpha\partial_\xi^\beta\widetilde{u}(\xi), \ \widetilde{u} = \mathbf{F}u. \qquad (17.26)$$

17.15. Corollary. *$\mathbf{F}\mathcal{S} \subset \mathcal{S}$, i.e., $\mathbf{F}u \in \mathcal{S}$, if $u \in \mathcal{S}$.*

Proof. Since $u \in \mathcal{S}$, it follows that for any fixed $N \in \mathbb{N}$ and any $\alpha, \beta \in \mathbb{Z}_+^n$ there exists $d_{\alpha\beta} > 0$ such that $|\partial_x^\alpha(x^\beta u(x))| \leq d_{\alpha\beta}(1+|x|)^{-N}$. Therefore, by virtue of Lemma 17.14,

$$|\xi^\alpha\partial_\xi^\alpha\widetilde{u}(\xi)| \ \leq \|\mathbf{F}[\partial_x^\alpha(x^\beta u)]\|_C \leq D_{\alpha\beta}\sup_x|\partial_x^\alpha(x^\beta u)|. \qquad (17.27)$$

Thus, $\widetilde{u} \in \mathcal{S}$. □

17.16. Theorem. *The mappings $\mathbf{F} : \mathcal{S} \to \mathcal{S}$ and $\mathbf{F}^{-1} : \mathcal{S} \to \mathcal{S}$ are reciprocal continuous automorphisms of the space $\mathcal{S}$.*

Proof. $\mathbf{F}$ is linear and, by Theorem 17.6, monomorphic. Let us check that $\forall\widetilde{u} \in \mathcal{S}\ \exists u \in \mathcal{S}$ such that $\mathbf{F}u = \widetilde{u}$. Let $u_0 = \mathbf{F}\widetilde{u}$. Since $u_0 \in \mathcal{S}$, according to Theorem 17.6 $\widetilde{u} = \mathbf{F}^{-1}\mathbf{F}\widetilde{u} = \mathbf{F}^{-1}u_0$. Consider the function $u(x) = u_0(-x)$. We have $\widetilde{u} = \mathbf{F}^{-1}u_0 = \mathbf{F}u$. Definition 17.10 immediately implies that $\mathbf{F}u_j \to 0$ in $\mathcal{S}$, if $u_j \to 0$ in $\mathcal{S}$. The same arguments are valid for $\mathbf{F}^{-1}$. □

17.17. Lemma (the Parseval equality). *Let f and $g \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$(\mathbf{F}f, \mathbf{F}g)_{L^2} = (f, g)_{L^2}, \quad i.e., \quad \int\limits_{\mathbb{R}^n}\widetilde{f}(\xi)\bar{\widetilde{g}}(\xi)\,d\xi = \int\limits_{\mathbb{R}^n}f(x)\bar{g}(x)\,dx. \qquad (17.28)$$

Moreover,

$$\langle\mathbf{F}f, g\rangle = \langle f, \mathbf{F}g\rangle, \quad i.e., \quad \int\limits_{\mathbb{R}^n}\widetilde{f}(\xi)g(\xi)d\xi = \int\limits_{\mathbb{R}^n}f(x)\widetilde{g}(x)dx. \quad (17.29)$$

Proof. The Fubini theorem implies (17.29) since

$$\int\limits_{\mathbb{R}^n}f(x)\widetilde{g}(x)\,dx = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}f(x)e^{-\mathring{i}x\xi}g(\xi)\,dx\,d\xi \ = \int\limits_{\mathbb{R}^n}\widetilde{f}(\xi)g(\xi)\,d\xi$$

Let $h = \overline{\mathbf{F}g}$. Then $g = \overline{\mathbf{F}h}$, since

$$g(\xi) = (\mathbf{F}^{-1}\bar{h})(\xi) = \int e^{\mathring{i}x\xi}\bar{h}(x)dx = \overline{\int e^{-\mathring{i}x\xi}h(x)dx} = (\overline{\mathbf{F}h})(\xi).$$

Substituting $g(\xi) = \widetilde{\bar{h}}(\xi)$ and $\widetilde{g}(x) = \bar{h}(x)$ into (17.29), we obtain $(\mathbf{F}f, \mathbf{F}h)_{L^2} = (f, h)_{L^2}$ $\forall h \in \mathcal{S}$, i.e., (up to the notation) (17.28). $\square$

Note that both sides of equality (17.29) define linear continuous functionals on $\mathcal{S}$:

$$f : \mathcal{S} \ni \widetilde{g} \longmapsto \int f(x)\widetilde{g}(x)\,dx, \ \ \widetilde{f} : \mathcal{S} \ni g \longmapsto \int \widetilde{f}(\xi)g(\xi)\,d\xi.$$

In this connection, we give (following L. Schwartz [**54**]) two definitions.

17.18. Definition. $\mathcal{S}'(\mathbb{R}^n)$ is the *space of tempered distributions*, i.e., the space of linear continuous functionals $f : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ equipped with the operation of differentiation

$$\langle \partial^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle f, \partial^\alpha \varphi \rangle,$$

where $\alpha \in \mathbb{Z}^n_+$, and with the operation of multiplication $\langle af, \varphi \rangle = \langle f, a\varphi \rangle$ by any *tempered function* a, i.e., a function $a \in C^\infty(\mathbb{R}^n)$ satisfying the condition $\forall \alpha \ \exists C_\alpha < \infty \ \exists N_\alpha < \infty$, such that $|\partial^\alpha a(x)| \le C_\alpha (1 + |x|)^{N_\alpha}$.

17.19. Definition. Let $f \in \mathcal{S}'$, $g \in \mathcal{S}'$. Then the formulae

$$\langle \mathbf{F}f, \varphi \rangle = \langle f, \mathbf{F}\varphi \rangle \ \forall \varphi \in \mathcal{S} \ \text{ and } \ \langle \mathbf{F}^{-1}g, \psi \rangle = \langle g, \mathbf{F}^{-1}\psi \rangle \ \forall \psi \in \mathcal{S} \tag{17.30}$$

specify the generalized functions $\widetilde{f} = \mathbf{F}f \in \mathcal{S}'$ and $\mathbf{F}^{-1}g \in \mathcal{S}'$, which are called the *Fourier transform of the distribution* $f \in \mathcal{S}'$ and the *inverse Fourier transform of the distribution* $g \in \mathcal{S}'$.

17.20. Example. It is clear that $\delta \in \mathcal{S}'$, $1 \in \mathcal{S}'$. Find $\mathbf{F}\delta$ and $\mathbf{F}1$. We have

$$\langle \mathbf{F}\delta, \varphi \rangle = \langle \delta, \mathbf{F}\varphi \rangle = \widetilde{\varphi}(0) = \lim_{\xi \to 0} \int e^{-\mathring{\imath} x\xi} \varphi(x)dx = \int \varphi(x)dx = \langle 1, \varphi \rangle,$$

i.e., $\mathbf{F}\delta = 1$. Similarly, $\mathbf{F}^{-1}\delta = 1$. Furthermore, $\langle \mathbf{F}1, \varphi \rangle = \langle 1, \mathbf{F}\varphi \rangle = \langle \mathbf{F}^{-1}\delta, \mathbf{F}\varphi \rangle = \langle \delta, \mathbf{F}^{-1}\mathbf{F}\varphi \rangle$, i.e., $\mathbf{F}1 = \delta$. Similarly, $\mathbf{F}^{-1}1 = \delta$.

17.21.**P.** *Verify (compare with P.17.12) that* $\mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$.

17.22.**P.** *Prove (compare with P.16.19 and* [**57**]*) that* $f \in \mathcal{S}'(\mathbb{R}^n)$ *if and only if there exists a finite sequence* $\{f_\alpha\}_{|\alpha| \le N}$ *of functions* $f_\alpha \in C(\mathbb{R}^n)$ *satisfying the condition* $|f_\alpha(x)| \le C(1 + |x|)^m$, *where* $C < \infty$, $m < \infty$ *and such that* $f = \sum\limits_{|\alpha| \le N} \partial^\alpha f_\alpha$. *Thus,* $\mathcal{S}' \subset \mathcal{D}'$.

17.23.**P.** *Verify that the mappings* $\mathbf{F} : \mathcal{S}' \to \mathcal{S}'$ *and* $\mathbf{F}^{-1} : \mathcal{S}' \to \mathcal{S}'$ *are reciprocal automorphisms of the space* $\mathcal{S}'$ *which are continuous relatively the weak convergence in* $\mathcal{S}'$*, i.e, if* $\nu \to \infty$*, then* $\langle \mathbf{F} f_\nu, \varphi\rangle \to \langle \mathbf{F} f, \varphi\rangle\ \forall \varphi \in \mathcal{S} \Longleftrightarrow \langle f_\nu, \varphi\rangle \to \langle f, \varphi\rangle\ \forall\varphi \in \mathcal{S}$.

17.24.**P.** *Calculate* $\mathbf{F}\delta$ *and* $\mathbf{F}1$ *(compare with Example 17.20), by considering the sequence of functions*

$$\delta_\nu(x) = \nu \cdot 1_{[-1/\nu, 1/\nu]}(x) \quad \textit{and} \quad 1_\nu(x) = 1_{[-\nu,\nu]}(x),$$

where $1_{[a,b]} = \theta(x-a) - \theta(x-b)$.

17.25.**P.** *Considering the sequence of the functions*

$$f_\nu(x) = \theta_\pm(x) e^{\mp x/\nu}, \quad x \in \mathbb{R},$$

show (see (12.7)*) that* $\widetilde{\theta}_\pm(\xi) = \pm \dfrac{1}{\mathring{i}\xi \pm 0}$.

17.26. Remark. The space $\mathcal{S}'$ is complete with respect to the weak convergence, since $\mathcal{S}$ is a Frechét space (see P.17.11 and Remark 16.27).

17.27.**P.** *Show that formula* (17.26) *is valid for any* $u \in \mathcal{S}'$.

17.28. Lemma. *Let*[7] $f \in \mathcal{E}'(\mathbb{R}^n)$*. Then* $\widetilde{f} = \mathbf{F} f$ *is a tempered function (see Definition 17.18) and*

$$\widetilde{f}(\xi) = \langle f(x), e^{-\mathring{i}x\xi}\rangle. \tag{17.31}$$

[7] Recall that the space $\mathcal{E}'$ is defined in P.16.13.

Proof. By virtue of Theorem 16.20, $f = \sum_{|\alpha|\le m} \partial^\alpha f_\alpha$, where $f_\alpha \in C_0(\mathbb{R}^n)$. Therefore

$$\begin{aligned}
\langle \widetilde{f}(\xi), \varphi(\xi)\rangle &= \sum_\alpha \langle \partial_x^\alpha f_\alpha(x), (\mathbf{F}\varphi)(x)\rangle \\
&= \sum_\alpha (-1)^{|\alpha|} \langle f_\alpha(x), \partial_x^\alpha \widetilde{\varphi}(x)\rangle \\
&= \sum_\alpha (\mathring{i})^{|\alpha|} \langle f_\alpha(x), \mathbf{F}[\xi^\alpha \varphi(\xi)](x)\rangle \\
&= \sum_\alpha (\mathring{i})^{|\alpha|} \int f_\alpha(x) \left[\int e^{-\mathring{i}x\xi} \xi^\alpha \varphi(\xi) d\xi\right] dx.
\end{aligned}$$

Since $f_\alpha(x)e^{-\overset{\circ}{i}x\xi}\xi^\alpha\varphi(\xi) \in L^1(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, it follows (by the Fubini theorem) that

$$\begin{aligned}\langle \widetilde{f}(\xi), \varphi(\xi)\rangle &= \int \left[\int \sum_\alpha (\overset{\circ}{i})^{|\alpha|} f_\alpha(x) e^{\overset{\circ}{i}x\xi}\xi^\alpha dx\right]\varphi(\xi)d\xi \\ &= \int \langle \sum_\alpha f_\alpha(x), (\partial_x)^\alpha e^{-\overset{\circ}{i}x\xi}\rangle \varphi(\xi)d\xi \\ &= \int \langle f(x), e^{-\overset{\circ}{i}x\xi}\rangle\varphi(\xi)d\xi.\end{aligned}$$

Thus (by virtue of P.13.19), $\widetilde{f}(\xi) = \langle f(x), e^{-\overset{\circ}{i}x\xi}\rangle$. Similarly,

$$\partial^\beta \widetilde{f}(\xi) = \langle f(x), (-\overset{\circ}{i}x)^\beta e^{-\overset{\circ}{i}x\xi}\rangle. \tag{17.32}$$

Since $f \in \mathcal{E}' \subset \mathcal{S}'$, it follows that (by virtue of Definition 17.10) $\exists N \geq 1$ such that

$$|\langle f(x), \psi(x)\rangle| \leq N \sup_{x\in\mathbb{R}^n} \sum_{|\alpha|\leq N} (1+|x|)^N \cdot |\partial^\alpha\psi(x)| \ \forall \psi \in \mathcal{S}.$$

Therefore, $|\partial^\beta \widetilde{f}(\xi)| = |\langle f(x), \sigma(x)(-\overset{\circ}{i}x)^\beta e^{-\overset{\circ}{i}x\xi}\rangle| \leq C(1+|\xi|)^N$. □

18. The Fourier–Laplace transform. The Paley–Wiener theorem

Formula (17.26) proven in Section 17 (which is valid for $u \in S'$, see P.17.27) contains an important property of the Fourier transformation, which is often expressed in the following words: "after applying the Fourier transformation the derivation operator becomes the multiplication by the independent variable". More exactly, the following formula holds:

$$\mathbf{F}(D^\alpha_x u(x)) = \xi^\alpha \widetilde{u}(\xi), \tag{18.1}$$

where $D^\alpha_x = (\overset{\circ}{i})^{-|\alpha|}\partial^\alpha_x$, and $\widetilde{u} = \mathbf{F}u$, $u \in S'$.

Property (18.1) allows us to reduce, in a sense, problems involving linear differential equations to algebraic ones. Thus, applying the Fourier transformation to the differential equation

$$A(D_x)u(x) \equiv \sum_{|\alpha|\leq m} a_\alpha D^\alpha_x u(x) = f(x), \ a_\alpha \in \mathbb{C}, \ f \in \mathcal{S}', \tag{18.2}$$

we obtain an equivalent "algebraic" equation

$$A(\xi) \cdot \widetilde{u}(\xi) \equiv \left(\sum_{|\alpha| \le m} a_\alpha \xi^\alpha \right) \cdot \widetilde{u}(\xi) \ = \widetilde{f}(\xi), \ \widetilde{f} \in \mathcal{S}'. \tag{18.3}$$

As Hörmander [**29**] and Lojasiewicz [**41**] have proved, equation (18.3) has always a solution $\widetilde{u} \in \mathcal{S}'$ (see in this connection Remark 19.2). Therefore, the formula $u = \mathbf{F}^{-1}\widetilde{u}$ determines a solution of differential equation (18.2). Indeed, by virtue of (18.1) and Theorem 17.16,

$$f(x) = \mathbf{F}^{-1}(\widetilde{f}(\xi)) = \mathbf{F}^{-1}(A(\xi) \cdot \widetilde{u}(\xi)) \\ = A(D_x)\mathbf{F}^{-1}(\widetilde{u}(\xi)) = A(D_x)u(x).$$

This approach to construction of a solution of a linear differential equation with the help of the Fourier transformation is rather similar to the idea of the operational calculus[1)] (see, for instance, [**38**]) using the so-called Laplace transformation (introduced for the first time (see [**34**]) by Norwegian Niels Abel who was a mathematical genius, little-known while alive, and who died from consumption when he was 26 years old). The *Laplace transformation* transfers a function f of $t \in \mathbb{R}_+$ integrable with the "weight" e^{-st} for any $s > 0$, into the function

$$L[f](s) = \int_0^\infty e^{-st} f(t)dt, \ s > 0. \tag{18.4}$$

[1)] Let us illustrate the idea of the calculus of variations on the example of problem (6.11), restricting ourselves by the case $\sigma = 0$. In other words, we consider the problem

$$u_t = u_{xx}, \quad t > 0, \ |x| < 1; \ u\big|_{x=\pm 1} = 0; \ u\big|_{t=0} = 1. \tag{18.5}$$

As has been noted in Section 17, series (17.16) constructed by the Fourier method, which gives the solution of this problem, converges very slowly for small t. By the way, this can be foreseen, since the Fourier series converges slowly for discontinuous functions, and the function $u(x,t)$ is discontinuous at the angle points of the half-strip $\{|x| < 1, \ t > 0\}$. In this connection, consider preliminarily the problem

$$\frac{\partial T}{\partial \tau} = a\frac{\partial^2 T}{\partial \xi^2}, \ \xi > 0, \ \tau > 0; \ T\big|_{\xi=0} = T_1; \ T\big|_{\tau=0} = T_0, \tag{18.6}$$

which simulates the distribution of the temperature in the vicinity of an angle point.

Passing (see Section 6) to the dimensionless parameters in the standard way

$$r = \xi/\sqrt{a\tau},\ \ u = (T - T_1)/(T_0 - T_1),$$

from (18.6) we obtain that $u(\tau, \xi) = f(r)$, where the function f satisfies the following conditions:

$$f''(r) + \frac{r}{2} f'(r) = 0,\ \ f(0) = 0,\ \ f(\infty) = 1.$$

Hence, $u(\tau, \xi) = \operatorname{erf}(\xi/(2\sqrt{a\tau})) = 1 - \operatorname{erfc}(\xi/(2\sqrt{a\tau}))$, where

$$\operatorname{erf}(y) = \frac{2}{\sqrt{\pi}} \int\limits_0^y e^{-\eta^2} d\eta,\ \ \operatorname{erfc}(y) = 1 - \operatorname{erf}(y).$$

These preliminary arguments suggest that the solution $u(x,t)$ of problem (18.5)for small t, seemingly, must be well approximated by the following sum

$$1 - [\operatorname{erfc}((1-x)/2\sqrt{t}) + \operatorname{erfc}((1+x)/2\sqrt{t})]. \tag{18.7}$$

This point allows us to obtain the representation of the solution of problem (18.5) in the form of a series rapidly converging for small t with the help of the Laplace transformation. Denoting the function $L[u(\cdot, x)](s)$ by $v(s,x)$, where u is the solution of problem (18.5), we rewrite, taking into account the two following obvious properties of the Laplace transformations:

$$L[1](s) = 1/s,\ \ L[f'](s) = s \cdot L[f](s) - f(0), \tag{18.8}$$

problem (18.5) in the form ("algebraic" in the variable s)

$$(s \cdot v(s,x) - 1) - v_{xx}(s,x) = 0,\ \ v(s,x)\big|_{x=\pm 1} = 0,\ \ s > 0.$$

This problem can be solved explicitly. Obviously, its solution is the function

$$v(s,x) = \frac{1}{s} - \frac{1}{s} \cdot \operatorname{ch}(\sqrt{s}x)/\operatorname{ch}(\sqrt{s}).$$

Thus, the solution u of problem (18.5) satisfies the relation

$$L[u(\cdot,x)](s) = \frac{1}{s} - \frac{1}{s} \cdot \operatorname{ch}(\sqrt{s}x)/\operatorname{ch}(\sqrt{s}). \tag{18.9}$$

Formulae (18.7) and (18.9) suggest that in order to obtain the representation of the solution of problem (18.5) in the form of a series rapidly converging for small t we should

- first, find the Laplace transform of the function $f_k(t) = \operatorname{erfc}(k/2\sqrt{t})$;
- second, represent the right-hand side of formula (18.9) in the form of a series whose members are function of the form $L[f_k]$.

Below, it will be shown that

$$L[f_k](s) = \frac{1}{s}\exp(-k\sqrt{s}). \tag{18.10}$$

On the other hand, expressing ch via exp and representing $(1+q)^{-1}$, where $q = \exp(-2\sqrt{s}) < 1$, as the series $1 - q + q^2 - q^3 + \cdots$, we obtain that

$$-\frac{\operatorname{ch}(\sqrt{s}x)}{s \cdot \operatorname{ch}\sqrt{s}} = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{\exp[-\sqrt{s}(2n+1-x)] + \exp[-\sqrt{s}(2n+1+x)]}{s}. \tag{18.11}$$

Basing on (18.8)–(18.11), one can show that the solution of problem (18.5) can be represented as a series

$$u(x,t) = 1 + \left[\sum_{n=0}^{N}(-1)^{n+1}a_n\right] + r_N, \tag{18.12}$$

where $a_n = \operatorname{erfc}((2n+1-x)/2\sqrt{t}) + \operatorname{erfc}((2n+1+x)/2\sqrt{t})$ and $r_N = \sum\limits_{n>N}(-1)^{n+1}a_n$.

18.1.**P.** *Show that in* (18.12)

$$|r_N| \le \frac{2}{N}\sqrt{\frac{t}{\pi}}\exp(-N^2/t). \tag{18.13}$$

18.2.**P.** *By comparing estimate* (18.13) *with estimate* (17.19)*, show that, for* $t \le 1/4$*, it is more convenient to use the representation of the solution of problem* (18.5) *in form* (18.12)*, but for* $t \ge 1/4$*, it is better to use that form* (17.16)*.*

Let us establish formula (18.10). It follows from formula

$$L[f_k'](s) = \exp(-k\sqrt{s}) \tag{18.14}$$

and the second formula in (18.8), since

$$f_k(0) = 0, \qquad f_k'(t) = \frac{k}{2}\pi^{-1/2}t^{-3/2}\exp(-k^2/4t).$$

As for formula (18.14), it can be proved, taking into account P.18.3, in the following way

$$L[f_k'](s) = \frac{k}{2\sqrt{\pi}} \int_0^\infty t^{-3/2} \cdot e^{-k^2/4t}\, dt = \frac{2}{\sqrt{\pi}} \int_0^\infty \exp -[\eta^2 + \frac{k^2 s}{4\eta^2}] d\eta$$

$$= \frac{2}{\pi} e^{-k\sqrt{s}} \int_0^\infty e^{-(\eta - a/\eta)^2} d\eta = e^{-k\sqrt{s}}.$$

(The change of variables: $\eta = \dfrac{k}{2\sqrt{t}}$, $a = \dfrac{k}{2}\sqrt{s}$.)

18.3. **P.** *Let* $F(a) = \int_0^\infty \exp[-(\eta - \frac{a}{\eta})^2] d\eta$, *where* $a > 0$. *Then* $F \equiv \frac{\sqrt{\pi}}{2}$.

HINT. $F'(a) \equiv 0$.

There is a close connection between the Fourier and Laplace transformations. It can be found, by analyzing the equality

$$\partial^\beta \widetilde{f}(\xi) = \langle f(x), (-\mathring{i}x)^\beta e^{-\mathring{i}x\xi} \rangle, \ f' \in \mathcal{E}'(\mathbb{R}^n), \ \beta \in \mathbb{Z}_+^n,$$

which has been proved in Lemma 17.28. The right-hand side of this equality is meaningful for any complex $\xi \in \mathbb{C}^n$ and is a continuous function in $\mathbb{C}^n$. Thus, as is known from the theory of functions of a complex variable, the analytic function is defined

$$\widetilde{f} : \mathbb{C}^n \ni \xi \longmapsto \widetilde{f}(\xi) = \langle f(x), e^{-\mathring{i}x\xi} \rangle \in \mathbb{C},$$

which can be treated as the *Fourier transform in the complex domain.* This function is sometimes called the *Fourier–Laplace transform.* This name can be justified by the fact that, for instance, for the function $f = \theta_+ f \in L^1(\mathbb{R})$ (compare with Example 17.3), the function

$$\mathbb{C}_- \ni \xi \longmapsto \int_{-\infty}^\infty e^{-\mathring{i}x\xi} f(x) dx = \int_0^\infty e^{-\mathring{i}x\xi} f(x) dx \in \mathbb{C},$$

being analytic in the lower half-plane $\mathbb{C}_-$, is, for real ξ (respectively, for imaginary $\xi = -is/2\pi$, where $s > 0$), the Fourier transform (respectively, the Laplace transform) of the function f.

The important role of the Fourier–Laplace transformation consists in the fact that, due to so-called Paley–Wiener theorems, certain properties of this analytic function allow to determine whether this function is the Fourier–Laplace transform of a function f as well as to characterize the properties of this function f. In Section 22 we shall use (compare with Example 17.3)

18.4. Theorem (Paley–Wiener). *Let $\widetilde{f}$ be analytic in $\mathbb{C}_-$ and*

$$\sup_{\eta>0} \int\limits_{-\infty}^{\infty} |\widetilde{f}(\xi - i\eta)|^2 d\xi < \infty.$$

Then $\widetilde{f}$ is the Fourier transform in $\mathbb{C}_-$ of the function $f = \theta_+ f \in L^2(\mathbb{R})$.

The proof of this theorem and of the inverse one see in [**71**].

19. Fundamental solutions. Convolution

At the beginning of Section 18 it was noted that the differential equation

$$A(D_x)u(x) \equiv \sum_{|\alpha|\le m} a_\alpha \partial_x^\alpha u(x) = f(x),\ a_\alpha \in \mathbb{C},\ f \in \mathcal{E}', \tag{19.1}$$

has a solution $u \in \mathcal{S}'$. In contrast to equation (18.2), the function f in (19.1) belongs to $\mathcal{E}' \subset \mathcal{S}'$. This fact allows us to give an “explicit” formula for the solution of equation (19.1), in which the role of the function f is emphasized. In this connection note that the formula

$$u(x) = \frac{1}{4\pi} \int\limits_{\mathbb{R}^3} f(y) \frac{\exp(-q|x-y|)}{|x-y|} dy,\ q \ge 0,\ f \in \mathcal{E}' \cap PC_b, \tag{19.2}$$

giving (compare with (7.8)–(7.9)) a solution of the equation $-\Delta u + q^2 u = f$, is meaningless for $q = 0$, if $\operatorname{supp} f$ is not compact (for instance, $f = 1$).

In order to deduce the desired “explicit” formula for the solution u of equation (19.1), first, we represent the function $\widetilde{u} = \mathbf{F}u$ in the form $\widetilde{u}(\xi) = \widetilde{f}(\xi)\widetilde{e}(\xi)$, where $\widetilde{e} \in \mathcal{S}'$ is a solution of the equation $A(\xi) \cdot \widetilde{e}(\xi) = 1$ (see Remark 19.2 below). Then, it remains to express the function $u = \mathbf{F}^{-1}(\widetilde{f} \cdot \widetilde{e})$ via $f = \mathbf{F}^{-1}\widetilde{f}$ and the function $e = \mathbf{F}^{-1}\widetilde{e}$ satisfying (by virtue of the relation $A(\xi)\widetilde{u}(\xi) \equiv 1$) the equation

$$A(D_x)e(x) = \delta(x). \tag{19.3}$$

19.1. Definition. A function $E \in \mathcal{D}'$ is called a *fundamental solution* of the operator $A(D_x)$, if $A(D_x)E(x) = \delta(x)$.

19.2. Remark. Any differential operator with constant coefficients has (as has been proved in [**29, 41**]) a fundamental solution in the class $\mathcal{S}'$. However, the presence of the space $\mathcal{D}'$ in Definition 19.1 is justified by the fact that, for some differential operators, it is possible (as has been shown by Hörmander) to construct in $\mathcal{D}'$ a fundamental solution locally more smooth than the fundamental solution in $\mathcal{S}'$. (Note that the two fundamental solutions E_1 and E_2 of the operator $A(D_x)$ differ by a function $v = E_1 - E_2$ satisfying the equation $A(D_x)v = 0$.)

If $A(\xi) \neq 0$ for any $\xi \in \mathbb{R}^n$, then the formula $E(x) = \mathbf{F}^{-1}(1/A(\xi))$, obviously, determines a fundamental solution of the operator $A(D_x)$. In this case, $E \in \mathcal{S}'$, because $1/A(\xi) \in \mathcal{S}'$. In the general case, the fundamental solution can be constructed, for instance, with the help of regularization of the integral $\int \widetilde{\varphi}(\xi)d\xi/A(\xi)$ (compare with P.14.4) that can be most simply made for $\varphi \in \mathcal{D}$, since in this case the regularization (by virtue of analyticity of the function $\widetilde{\varphi} = \mathbf{F}\varphi$) is possible by passing of ξ into the complex domain, where $A(\xi) \neq 0$ (see, for instance, [**57**]).

19.3. Examples. It follows from P.7.1 that function (7.9) is the fundamental solution of the Laplace operator. According to P.12.8, the function $\theta(t - |x|)/2$ is the fundamental solution of the string operator. For the heat operator $\partial_t - \partial_{xx}$, the fundamental solution is $E(x,t) = \theta(t)P(x,t)$, where the function P is defined in (6.15). Indeed, by virtue of the properties of the function P proven in Section 6, for any $\varphi \in \mathcal{D}(\mathbb{R}^2)$, we have

$$\begin{aligned}\langle E_t - E_{xx}, \varphi\rangle &= -\langle E, \varphi_t + \varphi_{xx}\rangle = -\lim_{\epsilon\to+0} \int_\epsilon^\infty \int_{\mathbb{R}} (\varphi_t + \varphi_{xx})E\,dx\,dt \\ &= \lim_{\epsilon\to+0}\left[\int_\epsilon^\infty \int_{\mathbb{R}} (P_t - P_{xx})\varphi\,dx\,dt + \int_{-\infty}^{\infty} P(x,t)\varphi(x,t)\,dx\right] \\ &= \varphi(x,t)\big|_{x=t=0}.\end{aligned}$$

Before representing the solution $u = \mathbf{F}^{-1}(\widetilde{f}\cdot\widetilde{e})$ of equation (19.1) via $f \in \mathcal{E}'$ and the fundamental solution $e \in \mathcal{S}'$ of the operator

$A(D_x)$, we express the function $\mathbf{F}^{-1}(\widetilde{f} \cdot \widetilde{g})$ via f and g in the assumption that $g = \mathbf{F}^{-1}\widetilde{g} \in \mathcal{E}'$. According to Theorem 16.20, there exist multiindices α and β and continuous functions f_α and g_β with compact supports such that $f = D_x^\alpha f_\alpha,\ g = D_x^\beta g_\beta$. Therefore,

$$\mathbf{F}^{-1}(\widetilde{f}(\xi) \cdot \widetilde{g}(\xi)) = \mathbf{F}^{-1}(\xi^{\alpha+\beta}\widetilde{f}_\alpha(\xi) \cdot \widetilde{g}_\beta(\xi)) \ = D_x^{\alpha+\beta}\mathbf{F}^{-1}(\widetilde{f}_\alpha \cdot \widetilde{g}_\beta).$$

Let us calculate $\mathbf{F}^{-1}(\widetilde{f}_\alpha \cdot \widetilde{g}_\beta)$. We have

$$\widetilde{f}_\alpha \cdot \widetilde{g}_\beta = \iint e^{-\mathring{i}(y+\sigma)\xi} f_\alpha(y) g_\beta(\sigma)\, dy\, d\sigma.$$

Setting $y+\sigma = x$ and using the compactness of $\operatorname{supp} f_\alpha$ and $\operatorname{supp} g_\beta$, we obtain $\widetilde{f}_\alpha \cdot \widetilde{g}_\beta = \mathbf{F}[f_\alpha * g_\beta]$, where $\varphi * \psi$ denotes (see note 4 in Section 5) the *convolution* of the two functions φ and ψ, i.e.,

$$(\varphi * \psi)(x) = \int \varphi(y)\psi(x-y)dy.$$

19.4. **P.** *Verify that if* $\varphi \in C_0^{|\alpha|}$, $\psi \in C_0^{|\alpha|}$, *then* $\varphi * \psi = \psi * \varphi$, $D^\alpha(\varphi * \psi) = (D^\alpha \varphi) * \psi = \varphi * (D^\alpha \psi)$.

19.5. Definition. (Compare with P.19.4). The *convolution* $f*g$ of two generalized functions $f = D^\alpha f_\alpha \in \mathcal{E}'$ and $g = D^\beta g_\beta \in \mathcal{E}'$, where $f_\alpha,\ g_\beta \in C_0(\mathbb{R}^n)$, is the generalized function $D^{\alpha+\beta}(f_\alpha * g_\beta)$.

Note that for $f \in \mathcal{E}'$ and $g \in \mathcal{E}'$ the following formula holds:

$$\mathbf{F}^{-1}(\widetilde{f} \cdot \widetilde{g}) = f * g. \tag{19.4}$$

Indeed, $\mathbf{F}[D^{a+\beta}(f_a * g_\beta)] = \xi^{a+\beta}\mathbf{F}(f_a * g_\beta) = (\xi^a \widetilde{f}_a)(\xi^\beta \widetilde{g}_\beta)$.

Let us find $\mathbf{F}^{-1}(\widetilde{f} \cdot \widetilde{g})$ in the case when $f \in \mathcal{E}'$ and $g \in \mathcal{S}'$.

19.6. **P.** *Let* $f \in \mathcal{E}'$, $g \in \mathcal{S}'$. *Setting* $g_\nu(x) = \varphi(x/\nu)g(x)$, *where* $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\varphi \equiv 1$ *for* $|x| < 1$, *verify that* $g_\nu \in \mathcal{E}'$; $g_\nu \to g$ *in* $\mathcal{S}'$; $\widetilde{f} \cdot \widetilde{g}_\nu \to \widetilde{f} \cdot \widetilde{g}$ *in* $\mathcal{S}'$, *and* $\langle f * g_\nu, \varphi\rangle \to \langle \mathbf{F}^{-1}(\widetilde{f} \cdot \widetilde{g}), \varphi\rangle\ \forall \varphi \in \mathcal{S}$.

On the base of P.19.6, generalizing formula (19.4), we give

19.7. Definition. The *convolution* $f*g$ of two generalized functions $f \in \mathcal{E}'$ and $g \in \mathcal{S}'$ is the function from $\mathcal{S}'$, given by the formula $f * g = \mathbf{F}^{-1}(\widetilde{f} \cdot \widetilde{g})$. (Note that $\widetilde{f} \in C^\infty$, $\widetilde{g} \in \mathcal{S}'$.)

19.8. **P.** *(Compare with P.19.4).* $f \in \mathcal{E}'$, $g \in \mathcal{S}' \Longrightarrow f*g = g*f$; $D^\alpha(f*g) = (D^\alpha f) * g = f * (D^\alpha g)$; $\delta * g = g$.

It follows from abovesaid that the following theorem holds.

19.9. THEOREM (compare with (19.2) and note 4 in Sect. 5). *The desired "explicit" formula for the solution of equation* (19.1) *has the form* $u = f * e$, *where* $e \in \mathcal{S}'$ *is the fundamental solution of the operator* $A(D_x)$.

19.10.**P.** *Prove the Weierstrass theorem on uniform approximation of a continuous function* $f \in C(K)$ *by polynomials on a compact* $K \subset \mathbb{R}^n$: $\forall \epsilon > 0\ \exists$ *a polynomial* p *such that* $\|f(x) - p(x)\|_{C(K)} < \epsilon$.

HINT. Let Ω be a neighbourhood of K. Following the scheme of the proof of Lemma 13.10 and taking into account P.4.4, set

$$p(x) = \int_\Omega f(y)\delta_\nu(x-y)dy,$$

where

$$\delta_\nu(x) = \prod_{m=1}^{n}\left[\frac{\nu}{\sqrt{\pi}}(1-\frac{1}{\nu}x_m^2)^{\nu^3}\right], \quad x = (x_1, \ldots, x_n).$$

19.11. LEMMA. *Let* $u \in L^1 \cap C$, $v \in L^2$. *Then* $u * v \in L^2$ *and*

$$\|u * v\|_{L^2} \le \|u\|_{L^1} \cdot \|v\|_{L^2}. \tag{19.5}$$

PROOF. $\left|\int u(\xi-\eta)v(\eta)d\eta\right|^2 \le \left(\int |u(\xi-\eta)|d\eta\right) \cdot A(\xi) = \|u\|_{L^1} \cdot A(\xi)$, where $A(\xi) = \int |u(\xi-\eta)|\cdot|v(\eta)|^2 d\eta$. However, (see Lemma 8.26) $\int A(\xi)d\xi = \|v\|^2_{L^2} \cdot \|u\|_{L^1}$. □

20. On spaces H^s

The study of generalized solutions of equations of mathematical physics leads in a natural way to the family of Banach spaces $W^{p,m}$ introduced by Sobolev. Recall Definition 17.4 of the space $W^{p,m}(\Omega)$. For $p \ge 1$ and $m \in \mathbb{Z}_+$, the space $W^{p,m}(\Omega)$ is the Banach space of the functions $u \in L^p(\Omega)$ whose norm

$$\|u\|_{W^{p,m}(\Omega)} = \left(\int_\Omega \sum_{|\alpha|\le m} |\partial^\alpha u|^p dx\right)^{1/p} \tag{20.1}$$

is finite. Here, $\partial^\alpha u$ is the generalized derivative of the function u, i.e.,

$$\partial^\alpha u = v \in L^1_{loc}(\Omega) \quad \text{and}$$
$$\int_\Omega v \cdot \varphi dx = (-1)^{|\alpha|} \int_\Omega u \partial^\alpha \varphi dx \quad \forall \varphi \in C_0^\infty(\Omega). \tag{20.2}$$

S.L. Sobolev called the function v satisfying conditions (20.2) the *weak* derivative of order α of the function u. Maybe, this is the reason, why letter W appeared in the designation of the *Sobolev spaces*.

For $p = 2$, the spaces $W^{p,m}$ are Hilbert spaces (see note 4 in Section 17). They are denoted (apparently, in honour of Hilbert) by H^m. These spaces play a greatly important role in modern analysis. Their role in the theory of elliptic equations is outlined in Section 22. A detailed presentation of the theory of these spaces can be found, for instance, in [**8, 70**]; a brief one is given below.

20.1.**P.** *Using formula* (18.1), *verify that the space* $H^m(\mathbb{R}^n)$ *introduced above for* $m \in \mathbb{Z}_+$ *is the space of* $u \in \mathcal{S}'(\mathbb{R}^n)$ *such that* $(1 + |\xi|)^m (Fu)(\xi) \in L^2(\mathbb{R}^n)$.

20.2. Definition. Let $s \in \mathbb{R}$. The space $H^s = H^s(\mathbb{R}^n)$ consists of $u \in \mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ for which the norm

$$\|u\|_s = \|\langle\xi\rangle^s \cdot \widetilde{u}(\xi)\|_{L^2(\mathbb{R}^n)}, \quad \text{where} \quad \langle\xi\rangle = 1 + |\xi|, \ \widetilde{u} = Fu.$$

is finite.

20.3.**P.** *Verify that if* $\alpha > \beta$, *then* $\mathcal{S} \subset H^\alpha \subset H^\beta \subset \mathcal{S}'$ *and the operators of embedding are continuous and their images are dense.*

20.4. Theorem (Sobolev embedding theorem). *If* $s > n/2 + m$, *then the embedding* $H^s(\mathbb{R}^n) \subset C_b^m(\mathbb{R}^n)$ *holds and there exists* $C < \infty$ *such that*

$$\|u\|_{(m)} \leq C\|u\|_s \ \forall u \in H^s, \tag{20.3}$$

where

$$\|u\|_{(m)} = \sum_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^n} |\partial^\alpha u(x)|.$$

Proof. We have to prove that $\forall u \in H^s(\mathbb{R}^n) \ \exists v \in C_b^m(\mathbb{R}^n)$ such that $v = u$ almost everywhere and $\|v\|_{(m)} \leq C\|u\|_s$. It is sufficient to prove it for $m = 0$. The inequality

$$|u(\xi)| = \left| \int \widetilde{u}(\xi)\langle\xi\rangle^s \cdot \langle\xi\rangle^{-s} e^{\mathring{i}x\xi} d\xi \right| \le \|u\|_s \left(\int \langle\xi\rangle^{-2s} d\xi \right)^{1/2},$$

where $u \in \mathcal{S}(\mathbb{R})$, implies estimate (20.3). If $u \in H^s$, $u_n \in \mathcal{S}$ and $\|u_n - u\|_s \to 0$, then by virtue of (20.3) $\exists v \in C^0$ such that $\|u_n - v\|_{(0)} \to 0$ and $\|v\|_{(0)} \le C\|u\|_s$. Since $\|u-v\|_{L^2(\Omega)} \le C_\Omega(\|u-u_n\|_s + \|u_n - v\|_{(0)}) \to 0$, we have $u = v$ almost everywhere. $\square$

20.5.**P.** *Let $u(x) = \varphi(2x)\ln|\ln|x||$, where $x \in \mathbb{R}^2$ and φ is the function from Example 3.6. Show that $u \in H^1(\mathbb{R}^2)$. Thus, $H^{n/2}(\mathbb{R}^n)$ cannot be embedded into $C(\mathbb{R}^n)$.*

20.6.**P.** *Verify that $\delta \in H^s(\mathbb{R}^n)$ for $s < -n/2$.*

20.7. THEOREM (of Sobolev on traces). *Let $s > 1/2$. Then, for any (in general, discontinuous) function $u \in H^s(\mathbb{R}^n)$ the trace $\gamma u \in H^{s-1/2}(\mathbb{R}^{n-1})$ is defined, which coincides, for a continuous function u, with the restiction $u\big|_{x_n=0}$ of the function u onto the hyperplane $x_n = 0$. Moreover, $\exists C < \infty$ such that*

$$\|\gamma u\|'_{s-1/2} \le C\|u\|_s \quad \forall u \in H^s(\mathbb{R}^n), \tag{20.4}$$

where $\|\cdot\|'_\sigma$ is the norm in the space $H^\sigma(\mathbb{R}^{n-1})$.

PROOF. Let $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. For $u \in \mathcal{S}(\mathbb{R}^n)$, we have $u(x', 0) = \int_{\mathbb{R}^{n-1}} e^{\mathring{i}x'\xi'} \left[\int_{\mathbb{R}} \widetilde{u}(\xi', \xi_n) d\xi_n\right] d\xi'$; therefore, $(\|\gamma u\|'_{s-1/2})^2 = \int_{\mathbb{R}^{n-1}} \langle\xi'\rangle^{2s-1} \left|\int_{\mathbb{R}} \widetilde{u}(\xi', \xi_n) d\xi_n\right|^2 d\xi'$. Then

$$\left| \int \widetilde{u}(\xi', \xi_n) d\xi_n \right|^2 \le A(\xi') \int \langle\xi\rangle^{2s} |\widetilde{u}(\xi)|^2 d\xi_n,$$

where $A(\xi') = \int \langle\xi\rangle^{-2s} d\xi_n \le C_s \langle\xi'\rangle^{-s+1/2}$, and $C_s = C\int(1 + z^2)^{-s} dz < \infty$ for $s > 1/2$; ($z = \xi_n(1 + |\xi'|^2)^{-1/2}$). Therefore, $\|\gamma u\|'_{s-1/2} \le C\|u\|_s$ for $u \in \mathcal{S}$. If $u \in H^s(\mathbb{R}^n)$ and $\|u_n - u\|_s \to 0$ as $n \to \infty$ and $u_n \in \mathcal{S}$, then $\exists w \in H^{s-1/2}(\mathbb{R}^{n-1})$ such that $\|\gamma u_n - w\|'_{s-1/2} \to 0$; w does not depend on the choice of the sequence $\{u_n\}$. By definition, $\gamma u = w$; and estimate (20.4) is also valid. $\square$

20.8. DEFINITION. The operator $P = P_\Omega : \mathcal{D}'(\mathbb{R}^n) \ni f \longmapsto Pf \in \mathcal{D}'(\Omega)$, where Ω is a domain in $\mathbb{R}^n$ and $\langle Pf, \varphi\rangle = \langle f, \varphi\rangle\ \forall \varphi \in$

$\mathcal{D}(\Omega)$, is called the *restriction operator* of generalized functions given in $\mathbb{R}^n$ onto the domain Ω.

20.9. Definition. Let $H^s(\Omega)$ denote the space $P_\Omega H^s(\mathbb{R}^n)$ with the norm
$$\|f\|_{s,\Omega} = \inf \|Lf\|_s, \quad f \in H^s(\Omega), \tag{20.5}$$
where the infimum is taken over all continuations $Lf \in H^s(\mathbb{R}^n)$ of the function $f \in H^s(\Omega)$ (i.e., $P_\Omega Lf = f$). If it is clear from the context that a function $f \in H^s(\Omega)$ is considered, then we may omit index Ω write $\|f\|_s$ and instead of $\|f\|_{s,\Omega}$.

20.10. Definition. The space $H^s(\Gamma)$, where $\Gamma = \partial\Omega$ is the smooth boundary of a domain $\Omega \Subset \mathbb{R}^n$, is the completion of the space $C^\infty(\Gamma)$ in the norm
$$\|\rho\|'_{s,\Gamma} = \sum_{k=1}^{K} \|\varphi_k \rho\|'_s. \tag{20.6}$$
Here, $\|\cdot\|'_s$ is the norm of the space $H^s(\mathbb{R}^{n-1})$, $\sum\limits_{k=1}^{K} \varphi_k \equiv 1$ is the partition of unity (see Section 3) subordinate to the finite cover $\bigcup\limits_{k=1}^{K} \Gamma_k = \Gamma$, where $\Gamma_k = \Omega_k \cap \Gamma$, and Ω_k is a n-dimensional domain, in which the normal to Γ do not intersect. Furthermore, the function $\varphi_k\rho \in C_0^\infty(\mathbb{R}^{n-1})$ is defined by the formula $(\varphi_k\rho)(y') = \varphi_k(\sigma_k^{-1}(y')) \cdot \rho(\sigma_k^{-1}(y'))$, where σ_k is a diffeomorphism of $\mathbb{R}^n$, (affine outside a ball and) "unbending" Γ_k. This means that, for $x \in \Omega_k$, the nth coordinate $y_n = y_n(x)$ of the point $y = (y', y_n) = \sigma_k(x)$ is equal to the coordinate of this point on the inward normal to Γ. If it is clear from the context that we deal with a function $\rho \in H^s(\Gamma)$, then equally with $\|\rho\|'_{s,\Gamma}$ we also write $\|\rho\|'_s$.

20.11. Remark. Definition 20.10 of the space $H^s(\Gamma)$ is correct, i.e., it does not depend on the choice of the cover, the partition of unity and the diffeomarphism σ_k. In the book [**59**] this fact is elegantly proven with the help of the technics of pseudodifferential operators.

20.12.**P.** *Let $s > 1/2$. Show that the operator $C(\bar{\Omega}) \cap H^s(\Omega) \ni u \longmapsto u\big|_\Gamma \in C(\Gamma)$ can be continued to a continuous operator $\gamma : H^s(\Omega) \to H^{s-1/2}(\Gamma)$.*

20.13. REMARK. The function $\gamma u \in H^{s-1/2}(\Gamma)$, where $s > 1/2$, is called the boundary value of the function $u \in H^s(\Omega)$. One can rather easily show (see, for instance, [**70**]) that $H^{s-1/2}(\Gamma)$, where $s > 1/2$, is the space of boundary values of functions from $H^s(\Omega)$. The condition $s > 1/2$ is essential, as the example of the function $u \in H^{1/2}(\mathbb{R}_+)$ given (compare with P.20.5) by the formula $u(x) = \varphi(2x) \ln |\ln |x||$ shows.

20.14. REMARK. The known *Arzelà theorem* (see [**36, 71**]) asserts that if a family $\{f_n\}$ of functions $f_n \in C(\bar{\Omega})$ defined in $\Omega \Subset \mathbb{R}^n$ is uniformly bounded (i.e., $\sup\limits_n \|f_n\| < \infty$) and equicontinuous ($\forall \epsilon > 0\ \exists \delta > 0$ such that $|f_n(x) - f_n(y)| < \epsilon\ \forall n$, if $|x-y| < \delta$), then one can choose from this sequence a subsequence converging in $C(\bar{\Omega})$. With the help of this theorem one can prove (see, for instance, [**8, 70**]) that the following theorem is valid.

20.15. THEOREM (on compactness of the embedding). *Let $\Omega \Subset \mathbb{R}^n$, and a sequence $\{u_n\}$ of functions $u_n \in H^s(\Omega)$ (respectively, $u_n \in H^s(\partial\Omega)$) be such that $\|u_n\|_s \leq 1$ (respectively, $\|u_n\|'_s \leq 1$). Then one can choose from this sequence a subsequence converging in $H^t(\Omega)$ (respectively, in $H^t(\partial\Omega)$), if $t < s$.*

21. On pseudodifferential operators (PDO)

The class of PDO is more large than the class of differential operators. It includes the operators of the form

$$Au(x) = \int\limits_\Omega K(x, x-y)u(y)dy,\ u \in C_0^\infty(\Omega).$$

Here, $K \in \mathcal{D}'(\Omega \times \mathbb{R}^n)$ and $K \in C^\infty(\Omega \times (\mathbb{R}^n \setminus 0))$. If $K(x, x-y) = \sum\limits_{|\alpha| \leq m} a_\alpha(x) \cdot \delta^{(\alpha)}(x-y)$, then $Au(x) = \sum\limits_{|\alpha| \leq m} a^\alpha(x)\partial^\alpha u(x)$. Another example of PDO is the singular integral operators [**45**]. However, the exclusive role of the theory of PDO in modern mathematical physics (which took the shape in the middle sixties [**30, 35, 66**]) is determined even not by the specific important examples. The matter is that PDO is a powerful and convenient tool for studying linear differential operators (first of all, elliptic one).

Before formulating the corresponding definitions and results, we would like to describe briefly the fundamental idea of the applications of PDO. Consider the elliptic differential equation in $\mathbb{R}^n$ with constant coefficients

$$a(D)u \equiv \sum_{|\alpha|\le m} a_\alpha D^\alpha u = f. \tag{21.1}$$

The ellipticity means that

$$a_m(\xi) \equiv \sum_{|\alpha|=m} a_\alpha \xi^\alpha \neq 0 \quad \text{for} \quad |\xi| \neq 0.$$

This is equivalent to the condition

$$|a(\xi)| \equiv \left| \sum_{|\alpha|\le m} a_\alpha \xi^\alpha \right| \ge C|\xi|^m \quad \text{for} \quad |\xi| \ge M \gg 1. \tag{21.2}$$

Now prove the following result on the smoothness of the solutions of equation (21.1). If $u \in H^{s-1}$ and $a(D)u \in H^{s-m}$ for some s, then $u \in H^s$. Of course, this fact can be established by constructing the fundamental solution of the operator and $a(D)$ and investigating its properties (see, for instance, [**31**]). However, instead of solving the difficult problem on mean value of the function $1/a(\xi)$, where $\xi \in \mathbb{R}^n$, (this problem arises after application the Fourier transformation to equation (21.1) written in the form $\mathbf{F}^{-1}a(\xi)\mathbf{F}u = f$) it is sufficient to note "only" two facts. The first one is that, taking into account (21.2), we can "cut out" the singularity of the function $1/a$ by a mollifier $\rho \in C^\infty$ such that $\rho \equiv 1$ for $|\xi| \ge M+1$, $\rho \equiv 0$ for $|\xi| \le M$. The second is that

$$(\mathbf{F}^{-1}(\rho/a)\mathbf{F})(\mathbf{F}^{-1}a\mathbf{F})u = u + (\mathbf{F}^{-1}\tau\mathbf{F})u, \;\; \tau = \rho - 1. \tag{21.3}$$

Therefore, by virtue of obvious inequalities

$$|\rho(\xi)/a(\xi)| \le C(1+|\xi|)^{-m}, \;\; |\tau(\xi)| \le C_N(1+|\xi|)^{-N} \; \forall N \ge 1, \tag{21.4}$$

which imply the inequalities

$$\|(\mathbf{F}^{-1}(\rho/a)\mathbf{F})f\|_s \le C\|f\|_{s-m}, \;\; \|(\mathbf{F}^{-1}\tau\mathbf{F})u\|_s \le C\|u\|_{s-N}, \tag{21.5}$$

as a result, we have a so-called *a priori estimate*

$$\|u\|_s \le C\left(\|f\|_{s-m} + \|u\|_{s-1}\right), \;\; f = a(D)u, \;\; u \in H^s, \tag{21.6}$$

where C does not depend on u. The above-mentioned result on the smoothness of the solution of elliptic equation (21.1) follows from

(21.6). Here the word “a priori” for estimate (21.6) of the solution of equation (21.1) means that the solution was obtained before the investigation of solvability of equation (21.1), i.e., *a priori*.

The simplicity of the deduction of the a priori estimate (21.6) characterizes brightly enough the role of the operators of the form $\mathbf{F}^{-1}a\mathbf{F}$. Such operators are called *pseudodifferential* operators constructed by the *symbol* $a = a(\xi)$. We shall also denote them by $Op(a(\xi))$ or $a(D)$. Depending on the class of symbols, one or another class of PDO is obtained. If $a(x,\xi) = \sum a_\alpha(x)\xi^\alpha$, then $a(x,D)u(x) = Op(a(x,\xi))u(x) = \sum a_\alpha(x)D_x^\alpha u(x)$. If $a(x,\xi)$ is a function positively homogeneous of zero order in ξ, i.e., $a(x,t\xi) = a(x,\xi)$ for $t > 0$, then $a(x,D) = Op(a(x,\xi))$ is the singular integral operator [**45**], namely,

$$Op(a(x,\xi))u(x) = b(x)u(x) + \lim_{\epsilon\to 0} \int\limits_{|x-y|>\epsilon} \frac{c(x,x-y)}{|x-y|^n} u(y)dy.$$

Here, $c(x,tz) = c(x,z)$ for $t > 0$ and $\int_{|z|=1} c(x,z)\,dz = 0$. In particular, in the one-dimensional case, when $a(\xi) = a_+\theta_+(\xi) + a_-\theta_-(\xi)$, where θ_+ is the Heaviside function and $\theta_- = 1 - \theta_+$, we have

$$Op(a(x,\xi))u = \frac{a_+ + a_-}{2\pi} u(x) + \frac{i}{2\pi}\text{v.p.}\int \frac{a_+ - a_-}{x-y} u(y)dy,$$

that follows from P.17.25 and (12.7).

Let us introduce a class of symbols important in PDO.

21.1. Definition. Let $m \in \mathbb{R}$. We denote by $S^m = S^m(\mathbb{R}^n)$ the class of functions $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that $a(x,\xi) = a_0(x,\xi) + a_1(\xi)$ and $\forall\alpha\ \forall\beta\ \exists C_{\alpha\beta} \in \mathcal{S}(\mathbb{R}^n)\ \exists C_\beta \in \mathbb{R}$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a_0(x,\xi)| \le C_{\alpha\beta}(x)\cdot\langle\xi\rangle^{m-|\beta|},$$
$$|\partial_\xi^\beta a_1(\xi)| \le C_\beta\langle\xi\rangle^{m-|\beta|} \quad, \langle\xi\rangle = 1 + |\xi|. \tag{21.7}$$

If $a \in S^m$, then the operator $a(x,D)$ given by the formula

$$Op(a(x,\xi))u(x) = \int e^{\mathring{i}x\xi} a(x,\xi)\widetilde{u}(\xi)d\xi,\ \widetilde{u} = Fu \tag{21.8}$$

and defined, obviously, on C_0^∞, can be continued to a continuous mapping from $H^s(\mathbb{R}^n)$ into $H^{s-m}(\mathbb{R}^n)$, as the following lemma shows.

21.2. Lemma (on continuity). *Let $a \in S^m$. Then $\forall s \in \mathbb{R}\ \exists C > 0$ such that*

$$\|a(x,D)u\|_{s-m} \le C\|u\|_s \quad \forall u \in C_0^\infty(\mathbb{R}^n). \tag{21.9}$$

Proof. If $a(x,\xi) = a_1(\xi)$, then estimate (21.9) is obvious. Therefore, it is sufficient to establish this estimate for $a = a_0$ (see Definition 21.1). Setting $A_0 v = a_0(x,D)v$, we note that

$$(\widetilde{A}_0 v)(\xi) = \int \left(\int e^{\mathring{i}(x,\xi-\eta)} a_0(x,\eta)dx \right) \widetilde{v}(\eta)d\eta.$$

By virtue of (21.1) and Lemma 17.14, we have

$$|(\widetilde{A_0 v})(\xi)| \le C_\alpha \int \langle\eta\rangle^m \langle\xi-\eta\rangle^{-|\alpha|} |\widetilde{v}(\eta)|d\eta, \ \ |\alpha| \gg 1.$$

The triangle inequality $|\xi| \le |\eta| + |\xi-\eta|$ implies the *Peetre inequality*

$$\langle\xi\rangle^s \le \langle\eta\rangle^s \langle\xi-\eta\rangle^{|s|}. \tag{21.10}$$

Therefore, $\langle\xi\rangle^{s-m}|(\widetilde{A}_0 v)(\xi)| \le C_\alpha \int \langle\eta\rangle^s \langle\xi-\eta\rangle^{|s-m|-|\alpha|}|\widetilde{v}(\eta)|d\eta$. It remains to apply inequality (19.5). □

21.3. Example. Let $a(\xi) = \epsilon + 1/(|\xi|^2 + q^2)$, $\epsilon \ge 0$, $q > 0$. Then $a \in S^0$ for $\epsilon > 0$ and $a \in S^{-2}$ for $\epsilon = 0$. Moreover, (compare with (19.2)) for $n = 3$, $a(D)u(x) = \epsilon u(x) + \pi \int_{\mathbb{R}^3} |x-y|^{-1} \exp(-2\pi q|x-y|)u(y)dy$, $u \in C_0^\infty$. Indeed, set $f = Op(1/(|\xi|^2 + q^2))u$ that is equivalent to $u = (|D|^2+q^2)f$, i.e., $-\Delta f + (2\pi q)^2 f = 4\pi^2 u$. By virtue of the estimate $\|f\|_s \le C\|u\|_{s+2}$, the solution of the last equation is unique in H^s. It can be represented in the form $f = 4\pi^2 G * u$, where (compare with P.7.1) $G(x) = \exp(-2\pi q|x|)/4\pi|x| \in H^0$ is the fundamental solution of the operator $-\Delta + (2\pi q)^2$.

21.4. Definition. Let $a \in S^m$. An operator $Op(a(x,\xi))$ is called *elliptic* if there exist $M > 0$ and $C > 0$ such that

$$|a(x,\xi)| \ge C|\xi|^m \ \forall x \in \mathbb{R}^n \ \text{ and } \ |\xi| \ge M$$

(compare with (21.2)).

21.5.**P.** *Following the above proof of estimate* (21.6) *and using Lemma 21.6, prove that, for the elliptic operator $Op(a(x,\xi))$ with the symbol $a \in S^m$ the prior estimate*

$$\|u\|_s \le C(\|a(x,D)\|_{s-m} + \|u\|_{s-N}) \ \forall u \in H^s, \ C = C(s,N), \ N \ge 1, \tag{21.11}$$

holds.

HINT. Setting $R = Op(\rho(\xi)/a(x,\xi))$, where $\rho \in C^\infty$, $\rho \equiv 1$ for $|\xi| \geq M+1$, $\rho \equiv 0$ for $|\xi| \leq M$, show that $R \cdot Op(a(x,\xi))u = u + Op(\tau(x,\xi))u$, where $\tau \in S^{m-1}$.

21.6. LEMMA (on composition). *Let $a \in S^k$, $b \in S^m$. Then $\forall N \geq 1$*

$$a(x,D) \cdot Op(b(x,\xi)) = \sum_{|\alpha|<N} Op\left[(\partial_\xi^\alpha a(x,\xi)(D_x^\alpha b(x,\xi)\right]/\alpha! + T_N,$$

where $\|T_N v\|_{s+N-(k+m)} \leq C\|v\|_s \ \forall v \in H^s$.

The proof see, for instance, in [**35**].

21.7. DEFINITION. An operator $T : C_0^\infty \to \mathcal{S}'$ is called *smoothing*, if $\forall N \geq 1\ \forall s \in \mathbb{R}\ \exists C > 0$ such that $\|Tu\|_{s+N} \leq C\|u\|_s\ \forall u \in H^s$.

21.8. **P.** *Let there be a sequence of functions $a_j \in S^{m_j}$, where $m_j \downarrow -\infty$ as $j \uparrow +\infty$. Then there exists a function $a \in S^{m_1}$ such that $(a - \sum\limits_{j<N} a_j) \in S^{m_N}\ \forall N > 1$.*

HINT. Following the idea of the proof of the Borel theorem 15.2, one can set

$$a(x,\xi) = \sum_{j=1}^{\infty} \varphi(\xi/t_j) a_j(x,\xi),$$

where $\varphi \in C^\infty(\mathbb{R}^n)$, $\varphi(\xi) = 0$ for $|\xi| \leq 1/2$, $\varphi(\xi) = 1$ for $|\xi| \geq 1$, and choose t_j tending to $+\infty$ as $j \to \infty$ so rapidly that, for $|x| \leq 1$ and $|\alpha| + |\beta| + 1 \leq j$, the following inequality holds:

$$\left|\partial_\xi^\alpha D_x^\beta (\varphi(\xi/t_j) a_j(x,\xi)\right| \leq 2^{-j} \langle\xi\rangle^{m_{j-1}}$$

Solution see, for instance, in [**59**].

21.9. DEFINITION. An operator $A : C_0^\infty \to \mathcal{S}'$ is called *pseudodifferential of class L*, if $A = Op(a(x,\xi)) + T$, where $a \in S^m$ for some $m \in \mathbb{R}$, and T is a smoothing operator. Any function $\sigma_A \in S^m$ such that $(\sigma_A - a) \in S^{-N}\ \forall N$ is called the *symbol* of the operator $A \in L$.

21.10. **P.** *Applying Lemma 21.6, show that the operator $A \in L$ has (compare with P.16.22) the pseudolocality property, in other words, if φ and ψ belong to C_0^∞ and $\psi = 1$ on $\operatorname{supp}\varphi$, then $\varphi A(1-\psi)$ is a smoothing operator.*

21.11. Remark. The class L is invariant with respect to the composition operation (see Lemma 21.6 and P.21.8) as well as with respect to the change of variables. The following lemma is valid (see, for instance, [**30, 59**]).

21.12. Lemma (on change of variables). *Let $a \in S^m$. Then the operator $a(x, D_x) = Op(a(x,\xi))$ in any coordinate system given by a diffeomorphism (affine outside a ball) $\sigma : x \longmapsto y = \sigma(x)$, for any $N \geq 1$, can be represented in the form*

$$\sum_{|\alpha|<N} Op\left[\varphi_\alpha(y,\eta)(\partial_\xi^\alpha a(x,\xi)\big|_{\xi={}^t\sigma'(x)\eta;\ x=\sigma^{-1}(y)}\right] + T_N, \qquad (21.12)$$

where ${}^t\sigma'(x)$ is the matrix transpose to $\sigma'(x) = \partial\sigma/\partial x$, and $\varphi_\alpha(y,\eta)$ is a polynomial in η of degree $\leq |\alpha|/2$ given by the formula

$$\varphi_\alpha(y,\eta) = \frac{1}{\alpha!} D_z^\alpha \exp\left[\mathring{i}(\sigma(z) - \sigma(x) - \sigma'(x)(h-x),\eta)\right]\big|_{z=x, x=\sigma^{-1}(y)}.$$

Moreover, $\|T_N v\|_{s+[N/2+1]-m} \leq C\|v\|_s \ \forall v \in H^s$.

22. On elliptic problems

In Section 5 we have considered (for some domains Ω) the simplest elliptic problem, namely, the Dirichlet problem for the Laplace equation. One can reduce to this problem the investigation of another important elliptic problem – the *problem with the directional* (the term *skew* is also used) *derivative* in a disk $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$ for the Laplace equation

$$\Delta u = 0 \ \text{ in } \ \Omega, \ \partial u/\partial\lambda = f \ \text{ on } \ \Gamma = \partial\Omega, \ f \in C^\infty(\Gamma). \qquad (22.1)$$

Here, $\partial/\partial\lambda = (a\partial/\partial x - b\partial/\partial y)$ is the differentiation along the direction λ (possibly, "sloping" with respect to the normal to the boundary Γ). This direction depends on the smooth vector field

$$\sigma : \Gamma \ni s \longmapsto \sigma(s) = (a(s), b(s)) \in \mathbb{R}^2, \ a^2(s) + b^2(s) \neq 0 \ \forall s \in \Gamma.$$

Let us identify a point $s \in \Gamma$ with its polar angle $\varphi \in [0, 2\pi]$. If $\sigma(\varphi) = (\cos\varphi, -\sin\varphi)$, then $\lambda = \nu$ is the (outward) normal to Γ; if $\sigma(\varphi) = -(\sin\varphi, \cos\varphi)$, then $\lambda = \tau$ is the tangent to Γ. All these cases as well as others are important in applications. However, our interest in problem (22.1) is caused, first of all, by the fact that it well illustrates the set of general elliptic problems.

It turns out that the solvability of problem (22.1) depends (see P.22.1–P.22.4 below) on the so-called *degree of the mapping* σ relatively the origin, namely, on the integer

$$N = \{\arg[a(2\pi) + ib(2\pi)] - \arg[a(0) + ib(0)]\}/2\pi.$$

It is clear that N is the number of revolutions with the sign, made by the point $\sigma(\varphi)$ around the origin, when it moves along the closed curve $\sigma : [0, 2\pi] \ni \varphi \longmapsto \sigma(\varphi) = (a(\varphi), b(\varphi))$.

For $N \geq 0$, problem (22.1) is always solvable but non-uniquely: the dimension α of the space of solutions of the homogeneous problem is equal to $2N + 2$.

If $N < 0$, then for solvability of problem (22.1), it is necessary and sufficient that the right-hand side f is "orthogonal" to a certain subspace of dimension $\beta = 2|N|-1$. More exactly, there exist $(2|N|-1)$ linear independent functions $\Phi_j \in L^2(\Gamma)$ such that problem (22.1) is solvable if and only if

$$\int_\Gamma f\Phi_j d\Gamma = 0 \ \forall j = 1, \dots, \beta = 2|N| - 1.$$

In this case the dimension α of the space of solutions of the homogeneous problem is equal to 1. A special case of problem (22.1), when $\lambda = \nu$ is the normal to Γ, is called the *Neuman problem* for the Laplace equation. In this case, $N = -1$, since $a(\varphi) + ib(\varphi) = \exp(-i\varphi)$. The Neuman problem is solvable if and only if $\int_\Gamma f d\Gamma = 0$; and the solution is defined up to an additive constant. Indeed, if $\int_\Gamma f d\Gamma = 0$, then a continuous function

$$g(s) = g(s_0) + \int_{s_0}^{s} f(\varphi) d\varphi.$$

is defined on Γ. Using the function g, we construct a solution v of the Dirichlet problem $\Delta v = 0$ in Ω, $v = g$ on Γ. Then the real part of the analytic function $u+iv$, i.e., the function u (defined up to an additive constant) is a solution of the considered Neuman problem, because $\partial u/\partial\nu = \partial v/\partial\tau = f$, where $\partial/\partial\tau$ is the differentiation along the tangent to Γ. Conversely, if u is a solution of the Neuman problem, then the "orthogonality" condition $\int_\Gamma f d\Gamma = 0$ holds. This follows[1)] immediately from the Gauss formula (7.5). Finally, the first Green formula (7.3) implies[1)] that if $u_1,\ u_2$ are two solutions of the Neuman problem, then $u = u_1 - u_2 = \text{const}$, since

$$\int\limits_{\Omega} (u_x^2 + u_y^2)\,dx\,dy = 0, \quad \text{i.e.,} \quad u_x = u_y \equiv 0 \Longleftrightarrow u = \text{const}\,.$$

[1)] Under the additional condition $u \in C^2(\bar{\Omega})$. This condition is valid for $f \in C^\infty(\Gamma)$ by virtue of prior estimates for elliptic problems (see below). By the way, the assertion is also valid without this additional condition (see, for instance, [**48**, § 28 and § 35]).

Now consider the general elliptic (see Definition 21.4) differential equation

$$a(x,D)u \equiv \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha u = f, \;\; u \in H^s(\Omega) \tag{22.2}$$

in a domain $\Omega \Subset \mathbb{R}^n$ with a smooth boundary Γ. The example of the problem with directional derivative shows that, first, it is advisable to ask the following question: how many boundary conditions

$$b_j(x,D)u\big|_\Gamma \equiv \sum_{|\beta| \le m_j} b_{j\beta}(x) D^\beta u\big|_\Gamma = g_j \quad \text{on} \quad \Gamma, \;\; j = 1, \dots, \mu \tag{22.3}$$

should be imposed (i.e., what is the value of the number μ) and what kind the boundary operators[2)] b_j should be of in order that the following two conditions hold:

1) problem (22.2)–(22.3) is solvable for any right-hand side

$$F = (f, g_1, \dots, g_\mu) \in H^{s,M} = H^{s,m}(\Omega) \times \prod_{j=1}^{\mu} H^{s-m_j-1/2}(\Gamma), \tag{22.4}$$

which is, possibly, orthogonal to a certain finite-dimensional subspace $Y_0 \subset H^{s,M}$;

2) the solution u of problem (22.2)–(22.3) is determined uniquely up to a finite-dimensional subspace $X_0 \subset H^s(\Omega)$.

[2)] The example of the problem $\Delta u = f$ in Ω, $\Delta u = g$ on Γ shows that one cannot assign arbitrary boundary operators (22.3).

Below we answer this question in terms of algebraic conditions on the leading terms of the symbols of differential operators. These conditions are sufficient as well as necessary (at least, for $n \ge 3$) for solvability of problem (22.2)–(22.3) in the above-mentioned sense. However, before considering these conditions, we formulate a series of exercises relating to the problem with directional derivative.

22.1. **P.** *Setting $U = u_x$, $V = -u_y$, show that a solution u of problem* (22.1) *determines the solution W of the following Hilbert problem: to find a function $W = U + iV$ analytic in Ω, continuous in $\bar{\Omega}$ and satisfying the boundary condition $aU + bV = f$ on Γ. Conversely, the solution W of this Hilbert problem determines uniquely, up to an additive constant, the solution u of problem* (22.1).

22.2. **P.** *Verify that on $\Gamma = \{z = |z|\exp(i\varphi),\ |z| = 1\}$ the continuous function $g(\varphi) = \arg[a(\varphi) + ib(\varphi)] - N\varphi$ is defined. Then, constructing in $\Omega = \{|z| < 1\}$ the analytic function $p + iq$ by the solution of the Dirichlet problem: $\Delta q = 0$ in Ω, $q = g$ on Γ, show that the function $c(z) = z^N \cdot \exp(p(x, y) + iq(x, y))$, where $z = x + iy$, analytic in Ω, satisfies on Γ the condition: $c = \rho \cdot (a + ib)$, where $\rho = e^p/|a + ib| > 0$.*

22.3. **P.** *Let $N \geq 0$ and $\zeta = \xi + i\eta$ be a function analytic in Ω and such that*

$$\Re\zeta = \rho \cdot f/|c|^2 \quad \text{on} \quad \Gamma. \tag{22.5}$$

Setting $U + iV = c(z)\zeta(z)$, verify that

$$\rho(aU + bV) = (\Re c)U + (\Im c)V = |c|^2\Re(U + iV)\zeta(z) = \rho f \quad \text{on} \quad \Gamma,$$

i.e., $(aU + bV) = f$ on Γ. Show that, for $N \geq 0$, the general solution of the Hilbert problem is representable in the form $c(z)[\zeta(z) + W_0(z)]$, where $W_0 = 0$ for $|z| = 1$, W_0 is analytic for $0 < |z| < 1$ and has a pole at $z = 0$ of multiplicity $\leq N$. Using Theorem 5.16, prove that

$$W_0(z) = i\mu_0 + \sum_{k=-N}^{-1} [(\lambda_k + i\mu_k)z^k - (\lambda_k - i\mu_k)z^{-k}],$$

where $\lambda_k \in \mathbb{R}$, $\mu_k \in \mathbb{R}$, i.e., $W_0(z)$ is a linear combination of $2N + 1$ linear independent functions.

22.4. **P.** *Let $N < 0$. Verify that if $U + iV$ is the solution of the Hilbert problem, then the function $\zeta(z) = (U + iV)/c(z)$ satisfies condition* (22.5). *Then, representing the function $\Re\zeta(z)$ harmonic for $|z| < 1$ in the form of the Poisson integral* (5.10) *and expanding the function $\Re\zeta(z)$ for $|z| = 1$ into the Fourier series,* *prove that* $\zeta(z) = (\lambda_0/2 + ic) + \sum\limits_{k=1}^{\infty}(\lambda_n - i\mu_n)z^n$ *for* $|z| < 1$*, where* $c \in \mathbb{R}$, $\lambda_n = \frac{1}{\pi}\int\limits_0^{2\pi} f(\varphi)\frac{\rho(\varphi)\cos(n\varphi)d\varphi}{a^2(\varphi)+b^2(\varphi)}$, $\mu_n = \frac{1}{\pi}\int\limits_0^{2\pi} f(\varphi)\frac{\rho(\varphi)\sin(n\varphi)d\varphi}{a^2(\varphi)+b^2(\varphi)}$. *Using this result and taking into account the fact that the function* $\zeta(z)$ *has at* $z = 0$ *zero of multiplicity* $\geq |N| \geq 1$*, show that, for* $N < 0$*, there exists at most one solution of the Hilbert problem, and the necessary and sufficient condition of the solvability of the Hilbert problem for* $N < 0$ *is the following one:* $\lambda_0 = \cdots = \lambda_{|N|-1} = \mu_1 = \cdots = \mu_{|N|-1} = 0$*, i.e., "orthogonality" of the function* f *to the* $(2|N| - 1)$*-dimensional space.*

22.5. Remark. The solution of problems P.22.1–P.22.4 is presented, for instance, in § 24 of the textbook [**25**].

Let us go back to the boundary-value problem (22.2)–(22.3) in the domain $\Omega \Subset \mathbb{R}^n$, representing it in the form of the equation $\mathcal{A}u = F$ for the operator

$$\mathcal{A} : H^s(\Omega) \ni u \longmapsto \mathcal{A}u \in H^{s,M}, \tag{22.6}$$

where $\mathcal{A}u \equiv (a(x,D)u, \gamma b_1(x,D)u, \dots, \gamma b_\mu(x,D)u)$. Here, γ is the operator of principal values on Γ (see P.20.12), and $H^{s,M}$ is the Banach space functions $F = (f, g_1, \dots, g_\mu)$ introduced in (22.4) with the norm

$$\|F\|_{s,M} = \|f\|_{s-m} + \sum_{j=1}^{\mu} \|g_j\|'_{s-m_j-1/2}. \tag{22.7}$$

Considering the operator equation $\mathcal{A}u = F$, we use the following standard notation. If X and Y are linear spaces and A a linear operator from X into Y, then

$$\operatorname{Ker} A = \{x \in X \mid Ax = 0\},\ \operatorname{Coker} A = Y/\operatorname{Im} A,$$

where $\operatorname{Im} A = \{y \in Y \mid y = Ax,\ x \in X\}$ is the image of the operator A, and $Y/\operatorname{Im} A$ is the factor-space of the space Y by $\operatorname{Im} A$, i.e., the linear space of the cosets with respect to $\operatorname{Im} A$ (see [**36**]). Recall that the linear spaces $\operatorname{Ker} A$ and $\operatorname{Coker} A$ are called the *kernel* and the *cokernel* of the operator A, respectively. Also recall that if X and Y are Banach spaces, then by $L(X, Y)$ we denote the space of linear continuous operators from X into Y.

As has been said, we are going to find conditions on the symbols of the operators $a(x,D)$ and $b_j(x,D)$ under which

$$\alpha = \dim \operatorname{Ker} \mathcal{A} < \infty, \ \beta = \dim \operatorname{Coker} \mathcal{A} < \infty. \tag{22.8}$$

Below we shall write property (22.8) briefly:

$$\operatorname{ind} \mathcal{A} = \alpha - \beta < \infty, \tag{22.9}$$

and call the number $\operatorname{ind} \mathcal{A} \in \mathbb{Z}$ by the *index* of the operator $\mathcal{A}$. Two most important results of the elliptic theory are connected with this concept: Theorem 22.23 on finiteness of the index and Theorem 22.27 on its stability.

22.6. Remark. It follows from P.22.7 and Lemma 22.8 that $\operatorname{Im} \mathcal{A}$ is closed in the Hilbert space $H^{s,M}$. Therefore, $\operatorname{Coker} \mathcal{A}$ is isomorphic to the orthogonal complement of $\operatorname{Im} \mathcal{A}$ in $H^{s,M}$.

22.7. **P.** *For any $s \in \mathbb{R}$, prove the existence of a continuous continuation operator*

$$\Phi : H^s(\Omega) \longrightarrow H^s(\mathbb{R}^n), \ P\Phi u = u \tag{22.10}$$

and show that operator (22.6) *is continuous for $s > \max_j(m_j) + 1/2$.*

Hint. For $\Omega = \mathbb{R}^n_+$, we can take as Φ the operator

$$\Phi f = Op(\langle \eta_- \rangle^{-s})\theta_+ Op(\langle \eta_- \rangle^{s}) Lf, \ \langle \eta_- \rangle = \eta_- + 1, \ \eta_- = -i\eta_n + |\eta'|, \tag{22.11}$$

where $L : H^s(\Omega) \to H^s(\mathbb{R}^n)$ is any continuation operator, and θ_+ is the characteristic function of $\mathbb{R}^n_+$. By virtue of the Paley–Wiener theorem 18.4, the function $\theta_+ Op(\langle \eta_- \rangle^s) Lf$ does not depend on L, since the function $\langle \eta_- \rangle^s (\widetilde{L_1 f} - \widetilde{L_2 f})$ can be analytically continued by η_n into $\mathbb{C}_+ \Rightarrow Op(\langle \eta_- \rangle^s)(L_1 f - L_2 f) = \theta_- g \in L^2$. Therefore, (see (20.5)) $\|\Phi f\|_{s,\mathbb{R}^n} \le C \inf_L \|\theta_+ Op(\langle \eta_- \rangle^s) Lf\|_{0,\mathbb{R}^n} \le C \inf_L \|Lf\|_{s,\mathbb{R}^n} = C\|f\|_{s,\mathbb{R}^n_+}$. If $\bar{\Omega}$ is a compact in $\mathbb{R}^n$, then $\Phi f = \varphi \cdot f + \sum_{k=0}^{K} \psi_k \cdot \Phi_k(\varphi_k \cdot f)$, where $\sum_{k=0}^{K} \varphi_k \equiv 1$ in Ω, $\varphi_k \in C_0^\infty(\Omega_k)$, and $\bigcup_{k=1}^{K} \Omega_k$ is a cover of the domain Ω such that $\bigcup_{k=1}^{K} \Omega_k \supset \Gamma$; $\psi_k \in C_0^\infty(\Omega_k)$, $\psi_k \varphi_k = \varphi_k$, and Φ_k is the operator given by formula (22.11) in the local coordinates $y = \sigma_k(x)$ "unbending" Γ (see Definition 20.10).

22.8. Lemma (L. Schwartz). *Let* $A \in L(X, Y)$ *and*

$$\dim \operatorname{Coker} A < \infty.$$

Then $\operatorname{Im} A$ *is closed in* Y.

Explanation. Consider an example. Let A be the operator of embedding of $X = C^1[0,1]$ into $Y = C[0,1]$. Obviously, $\operatorname{Im} A \neq Y = \overline{\operatorname{Im} A}$. According to Lemma 22.8, $\dim \operatorname{Coker} A = \infty$. This can readily be understood directly. Indeed, let $\varphi_\alpha(t) = |t - \alpha|$, where $\alpha \in]0,1[$, $t \in [0,1]$. We have $\varphi_\alpha \notin \operatorname{Im} A$, $\varphi_\alpha - \varphi_\beta \notin \operatorname{Im} A$ for $\alpha \neq \beta$, i.e., the elements φ_α are the representatives of linear independent vectors in $Y/\operatorname{Im} A$. Thus, $\dim(Y/\operatorname{Im} A) = \infty$.

The proof of Lemma 22.8 is based on the *Banach theorem* on the inverse operator[3)]. It is presented, for instance, in [**59**].

[3)] Let X and Y be Banach spaces, $A \in L(X, Y)$. If $\operatorname{Ker} A = 0$, then $\exists A^{-1} : \operatorname{Im} A \to X$. However, as the example given in the clarification to the lemma shows, the operator A^{-1} can be discontinuous. The *Banach theorem* (see [**36**]) asserts that A^{-1} is continuous, if $\operatorname{Im} A = Y$.

22.9. Lemma. *If* $\operatorname{ind} \mathcal{A} < \infty$*, then* $\exists C > 0$ *such that*

$$\|u\|_s \leq C(\|\mathcal{A}u\|_{s,M} + \|u\|_{s-1}) \ \forall u \in H^s(\Omega). \tag{22.12}$$

Proof. Let X_1 be the orthogonal complement in $H^{s,M}$ to $X_0 = \operatorname{Ker} \mathcal{A}$. We have $\mathcal{A} \in L(X_1, Y_1)$, where $Y_1 = \operatorname{Im} \mathcal{A}$, and $\mathcal{A}$ is an isomorphism of X_1 and Y_1. The space Y_1 is closed (Lemma 22.8), hence, it is a Banach space. By the Banach theorem, $\mathcal{A}^{-1} \in L(Y_1, X_1)$. Let p denote the orthogonal projector of X onto X_0. Then

$$\begin{aligned}
\|u\|_s &\leq \|pu\|_s + \|(1-p)u\|_s = \|pu\|_s + \|\mathcal{A}^{-1}\mathcal{A}(1-p)u\|_s \\
&\leq \|pu\|_s + C_1\|\mathcal{A}(1-p)u\|_{s,M} \\
&\leq \|pu\|_s + C_1\|\mathcal{A}u\|_{s,M} + C_2\|pu\|_s.
\end{aligned}$$

It remains to note that $\|pu\|_s \leq C\|u\|_{s-1}$. This is true, because $pu \in X_0$, $\dim X_0 < \infty$, hence, $\|pu\|_s \leq C\|pu\|_{s-1}$ (since the continuous function $\|v\|_s$ is bounded on the finite-dimensional sphere $\|v\|_{s-1} = 1$, $v \in X_0$). $\square$

22.10. Lemma. (22.12) $\Rightarrow \dim \operatorname{Ker} \mathcal{A} < \infty$.

PROOF. Suppose that $\dim \operatorname{Ker} \mathcal{A} = \infty$. Let $\{u_j\}_{j=1}^{\infty}$ be a orthonormed sequence in $X_0 = \operatorname{Ker} \mathcal{A}$. Then $\|u_k - u_m\|_s^2 = 2$. It follows from (22.12) that $\|u_k - u_m\|_s = \sqrt{2} \leq C\|u_k - u_m\|_{s-1}$, because $(u_k - u_m) \in X_0$. Therefore, $\|u_k - u_m\|_{s-1} \geq \sqrt{2}/C$. Hence, one cannot choose from the sequence $\{u_j\}$ bounded in $H^s(\Omega)$ a subsequence converging in $H^{s-1}(\Omega)$. However, this contradicts the compactness of the embedding of $H^s(\Omega)$ into $H^{s-1}(\Omega)$ (see Theorem 20.15). $\square$

22.11. REMARK. Lemmas 22.9 and 22.10 show the role of a priori estimate (22.12). A way to its proof is suggested by the proof of a priori estimate (21.11) in $\mathbb{R}^n$ (see hint to P.21.5). Moreover, the following lemma is valid.

22.12. LEMMA. *Let* $R \in L(H^{s,M}, H^s)$,

$$R \cdot \mathcal{A}u = u + Tu, \ \|Tu\|_{s+1} \leq C\|u\|_s \tag{22.13}$$

and

$$\mathcal{A} \cdot RF_1 = F_1 + T_1F_1, \ \|T_1F_1\|_{s+1,M} \leq C\|F_1\|_{s,M}. \tag{22.14}$$

Then $\operatorname{ind} \mathcal{A} < \infty$.

PROOF. Obviously, (22.13) $\Rightarrow$ (22.12). Therefore, $\dim \operatorname{Ker} \mathcal{A} < \infty$. Furthermore, $T_1 : H^{s,M} \to H^{s,M}$ is a *compact* (in other terms, *totally continuous*) operator [**36**], i.e., T_1 maps a bounded set in $H^{s,M}$ in a compact one that follows from (22.14) and the compactness of the embedding of $H^{s+1,M}$ into $H^{s,M}$ (Theorem 20.15). Hence, by the Fredholm theorem (see [**36, 56**]) the subspace $Y_1 = \operatorname{Im}(1 + T_1)$ is closed in $H^{s,M}$, where $\dim \operatorname{Coker} Y_1 < \infty$, and the equation $(1+T_1)F = F$ has a solution for any $F \in Y_1$. It remains to note that $\operatorname{Im} \mathcal{A} = \operatorname{Im}(1+T_1)$, and the equation $\mathcal{A}u = F$ has $\forall F \in Y_1$ the solution $u = RF_1$. $\square$

22.13. DEFINITION. The operator R, which satisfies (22.13) and (22.14), is called the *regulizer* of the operator $\mathcal{A}$.

22.14. **P.** *Let* $\Gamma = \partial\Omega$, *where* $\Omega \Subset \mathbb{R}^{n+1}$. *A pseudodifferential operator*[4)] $A : H^s(\Gamma) \to H^{s-m}(\Gamma)$ *of the class* L^m *on the closed variety* Γ *is called elliptic, if its symbol*[4)] a *satisfies the condition* $|a(x,\xi)| \geq C|\xi|^m$ *for* $x \in \Gamma$ *and* $|\xi| \gg 1$. *Prove that* $\operatorname{ind} A < \infty$.

[4)] Let $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ be the coordinate representation of a linear functional v on the tangent space to Γ at a point $p \in \Gamma$ with the local coordinates $x = (x_1, \dots, x_n)$. The functional (*vector*) v is called *cotangent*. The set of such vectors id denoted by $T_p^*\Gamma$. It is isomorphic to $\mathbb{R}^n$. The value of v on the tangent vector $\operatorname{grad}_x = (\partial/\partial x_1, \dots, \partial/\partial x_n)$ is calculated by the formula $(\xi, \operatorname{grad}_x) = \xi_1 \partial/\partial x_1 + \cdots + \xi_n \partial/\partial x_n$. If $y = \sigma(x)$ is another local system of coordinates of the same point $p \in \Gamma$ and $\eta = (\eta_1, \dots, \eta_n)$ is the corresponding coordinate representation for the cotangent vector v, then, by virtue of the equality $(\xi, \operatorname{grad}_x) = (\eta, \operatorname{grad}_y)$, the relation $\xi = {}^t\sigma'(x)\eta$ holds, where ${}^t\sigma'(x)$ is defined in Lemma 21.12. In the set $\bigcup\limits_{p\in\Gamma} T_p^*\Gamma$ a structure of a smooth variety is introduced in a natural way. It is called the *cotangent bundle*. Let a function $a \in C^\infty(T_p^*\Gamma)$ be such that, for the points of $\Gamma_k \subset \Gamma$ with local coordinates x, the function a coincides with a function $a_k \in S^m$. Let $\sum \varphi_k \equiv 1$ be a partition of unity subordinate to the cover $\cup\Gamma_k = \Gamma$, and $\psi_k \in C_0^\infty(\Gamma_k)$, where $\psi_k\varphi_k = \varphi_k$. Lemma 21.12 on the choice of variables implies that the formula $A : H^s(\Gamma) \ni u \longmapsto Au = \sum \varphi_k Op(a_k(x,\xi))\psi_k u \in H^{s-m}(\Gamma)$ uniquely, up to the operator $T \in L(H^s(\Gamma), H^{s-m+1}(\Gamma))$, determines a linear continuous operator that is called *pseudodifferential* of the class L^m with the *symbol* a.

Hint. Let $\sum \varphi_k \equiv 1$ be a partition of unity subordinate to a finite cover $\cup\Gamma_k = \Gamma$, and $\psi_k \in C_0^\infty(\Gamma_k)$, where $\psi_k\varphi_k = \varphi_k$. Show (compare with the hint to P.21.5), that the operator[4)]

$$Rf = \sum \psi_k Op(\rho_k(\xi)/a_k(x,\xi))\varphi_k f, \ f \in H^{s-m}(\Gamma), \tag{22.15}$$

where $\rho \in C^\infty(\mathbb{R}^n)$, $\rho = 1$ for $|\xi| \ge M+1$ and $\rho = 0$ for $|\xi| \le M$, is the regulizer for A.

We continue the investigation of the boundary-value problem (22.2)–(22.3). Always below we assume that the following condition holds.

22.15. Condition. *If* $\dim\Omega = 2$, *then the leading coefficients of the operator* $a(x,D)$ *are real.*

22.16. Lemma. *The principal symbol* $a_m(x,\xi) = \sum\limits_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ *of the operator* $a(x,D)$ *under Condition 22.15 always admits a factorization* [**66**], *i.e., the function*

$$a_m(y,\eta) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha\big|_{\xi={}^t\sigma'(x)\eta;\ x=\sigma^{-1}(y)},$$

where σ is defined in (21.12)*, can be represented in the form*

$$a_m(y,\eta) = a_+(y,\eta)\cdot a_-(x,\eta),\ \ \eta = (\eta', \eta_n) \in \mathbb{R}^{n-1}\times\mathbb{R}. \quad (22.16)$$

Here, the function $a_\pm(y,\eta)$ as well as the function $a_\pm^{-1}(y,\eta)$ is continuous for $\eta \neq 0$ and, for any $\eta' \neq 0$, can be analytically continued by η_n into the complex half-plane $\mathbb{C}_\mp$. In this case

$$a_\pm(y,t\eta) = t^\mu a_\pm(y,\eta) \ \ \textit{for}\ \ t>0,\ (\eta',\eta_n)\in\mathbb{R}^{n-1}\times\mathbb{C}_\mp,$$

where the number μ is integer[5]*. Moreover, $m = 2\mu$.*

[5] We shall see below that the number μ equal to the degree of homogeneity of the function $a_+(y,\eta)$ in η_n and called by the *index of factorization* of the symbol a_m is intendedly denoted by the same letter as the required number of boundary operators $b_j(x,D)$ in problem (22.2)–(22.3).

Proof. If the coefficients $a_\alpha(x)$ for $|\alpha| = m$ are real, then, for $\eta' \neq 0$, the equation $a_m(y,\eta) = 0$, where $\eta = (\eta',\eta_n) \in \mathbb{R}^{n-1}\times\mathbb{R}$, has with respect to η_n only complex-conjugate roots $\eta_n = \pm i\lambda_k(y,\eta') \in \mathbb{C}_\pm$, where $k = 1,\dots,\mu$. Therefore, $m = 2\mu$ is an even integer and

$$a_\pm(y,\eta) = c_\pm(y)\prod_{k=1}^{\mu}(\pm i\eta_n + i\lambda_k(y,\eta')),\ \ c_\pm(x)\neq 0. \quad (22.17)$$

For $n \geq 3$, formula (22.16) is always valid. Indeed, for $\eta_n \neq 0$, with every root $\eta_n = \pm i\lambda(y,\eta') \in \mathbb{C}_\pm$ of the equation $a_m(u,\eta) = 0$, where $\eta = (\eta',\eta_n)$, by virtue of the homogeneity of $a_m(u,\eta)$ with respect to η, the root $\eta_n = \mp\lambda(y,-\eta') \in \mathbb{C}_\mp$ is associated. It remains to note that the function $\lambda(y,\eta')$ is continuous with respect to $\eta' \neq 0$, and the sphere $|\eta'| = 1$ is connected for $n \geq 3$. □

22.17. Remark. It is clear that the symbol $|\eta|^2$ of the Laplace operator admits the factorization $|\eta|^2 = \eta_+\eta_-$, where $\eta_\pm = \pm i\eta_n + |\eta'|$. However, the symbol of the operator $(\partial/\partial y_2 + i\partial/\partial y_1)^m$ is not factorizable, since $(\eta_n + i\eta')^{-m}$ can be analytically continued with respect to η_n in $\mathbb{C}_+$ (in $\mathbb{C}_-$) only for $\eta' > 0$ ($\eta' < 0$).

Let us formulate the conditions imposed on the symbols of the operators $b_j(x,D)$. We fix a point $x_0 \in \Gamma$. Take the leading parts of the symbols of the operators $a(x_0,D)$ and $b_j(x_0,D)$, written in the coordinates $y = (y',y_n) \in \mathbb{R}^{n-1}\times\mathbb{R}$ which locally "unbend" Γ.

This means that near the point x_0 the boundary Γ is given by the equation $y_n = 0$, where y is the inward normal to Γ. Thus, consider the polynomials in η:

$$a_m(x_0, \eta) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha \quad \text{and}$$

$$b_{m_j}(x_0, \eta) = \sum_{|\beta|=m_j} b_{j\beta}(x_0)\xi^\beta, \quad j = 1, \dots, \frac{m}{2},$$

where (according to Lemma 21.12) $\xi = {}^t\sigma'(x_0)\eta$, and $\sigma : x \longmapsto y$ is the diffeomorphism "unbending" Γ near the point x_0. Let $\eta' \neq 0$. Suppose that Condition 22.15 holds, i.e., $a_m(x_0, \eta) = a_+(x_0, \eta) \cdot a_-(x_0, \eta)$. Let

$$\sum_{k=1}^{\mu} b_{jk}(\eta')\eta_n^k \equiv b_{m_j}(x_0, \eta) \mod a_+(x_0, \eta) \tag{22.18}$$

denote the remainder of division of $b_{m_j}(x_0, \eta)$ by $a_+(x_0, \eta)$ (where b_{m_j} and a_+ are considered as polynomials in η_n).

22.18. Condition. (of complementability [**2**], or the Shapiro–Lopatinsky condition [**3, 19**]). *Polynomials* (22.18) *are linear independent, i.e.,*

$$\det(b_{jk}(x, \eta')) \neq 0 \ \forall x \in \Gamma, \ \forall \eta' \neq 0. \tag{22.19}$$

In other words, the principal symbols $b_{m_j}(x, \eta)$ of the boundary operators, considered as polynomials in η_n, are linear independent modulo the function $a_+(x_0, \eta)$ which is a polynomial in η_n.

22.19. Remark. In the case of a differential operator $a(x, D)$ or in the case of a pseudodifferential operator $a(x, D)$ with a rational symbol, as in Example 21.3, we have $a_+(\eta', \eta_n) = (-1)^\mu a_-(-\eta', -\eta_n)$. Therefore, the function $a_+(x_0, \eta)$ in Condition 22.18 can be replaced by $a_-(x_0, \eta)$. By the same reason, in these cases it is unessential whether y_n is the inward normal or the outward normal to Γ.

22.20. Definition. Problem (22.2)–(22.3) and the corresponding operator $\mathcal{A}$ are called *elliptic*, if Conditions 22.15 and 22.18 hold.

22.21. Example. Let $a(x, D)$ be an elliptic operator of order $m = 2\mu$. Let $B_j(x, D) = \partial^{j-1}/\partial\nu^{j-1} + \dots$, $j = 1, \dots, \mu$, where ν is the normal to Γ, and dots denote an operator of order $< j - 1$. Then (under Condition 22.15) $\det(b_{jk}(x, \eta')) = 1$ for any $a(x, D)$.

22.22.**P.** *Let λ be a smooth vector field on $\Gamma = \partial\Omega$, where $\bar{\Omega}$ is a compact in $\mathbb{R}^n$. Show that the Poincaré problem*

$$a(x,D)u \equiv \sum_{|\alpha|\le 2} a_\alpha(x)D^\alpha u = f \quad in \quad \Omega, \ \partial u/\partial\lambda + b(x)u = g \quad on \quad \Gamma \tag{22.20}$$

for an elliptic operator $a(x,D)$ is elliptic in the case $n \ge 3$ if and only if the field λ at none point Γ is tangent to Γ. Verify also that in the case $n = 2$ problem (22.20) *is elliptic (under Condition 22.15) for any nondegenerate field λ.*

22.23. THEOREM. *Let the operator $\mathcal{A} : H^{s,M}(\Omega) \to H^s(\Omega)$ associated with the differential boundary-value problem* (22.2)–(22.3)*:*

$$a(x,D)u \equiv \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha u = f \quad in \quad \Omega \Subset \mathbb{R}^n,$$

$$b_j(x,D)u\big|_\Gamma \equiv \sum_{|\beta|\le m_j} b_{j\beta}(x)D^\beta u\big|_\Gamma = g_j \quad on \quad \Gamma, \ j = 1,\dots,\mu = m/2,$$

be elliptic. Let (compare with P.22.7) $s > \max\limits_j(m_j) + 1/2$. Then $\operatorname{ind}\mathcal{A} < \infty$*. Moreover,*

$$\|u\|_s \le C(\|a(x,D)u\|_{s-m} + \sum_{j=1}^{\mu} \|b_j(x,D)u\big|_\Gamma\|'_{s-m_j-1/2} + \|u\|_{s-1}). \tag{22.21}$$

PROOF. We outline the proof whose details can be found in [**2, 14, 66**]. Using the partition of unity (as has been suggested in hints to P22.7 and P.22.14) and taking into account P.21.5, we can reduce the problem of construction of the regulizer for the operator $\mathcal{A}$ to the case, when $\Omega = \mathbb{R}^n_+$, and the symbols $a(x,\xi)$ and $b_j(x,\xi)$ do not depend on x. In this case, we define the operator $R : H^{s,M} \to H^s$ by the formula

$$RF = P_+Op(r_+/a_+)\theta_+Op(r_-/a_-)Lf + \sum_{j=1}^{\mu} P_+Op(c_j)(g_j - f_j), \tag{22.22}$$

where P_+ is the operator of contraction on $\mathbb{R}^n_+$, $L : H^s(\mathbb{R}^n_+) \to H^s(\mathbb{R}^n)$ is any operator of continuation; by $r_\pm$ we denote the functions $\xi_\pm^\mu/\langle\xi_\pm\rangle^\mu$, "removing" the singularities of the symbols $1/a_\pm$ at the point $\xi = 0$, since $\xi_\pm = \pm i\xi_n + |\xi'|$, and $\langle\xi_\pm\rangle = \xi_\pm + 1$. Note that (in contrast to the analogical function ρ in (21.3)) the function $r_\pm$ can be analytically continued by $\xi_n \in \mathbb{C}_\mp$. Furthermore, $c_j(\xi) = \sum_{k=1}^{\mu} c_{jk}(\xi')(\xi_n^{k-1}/a_+(\xi))$, where $(c_{kj}(\xi'))$ is the inverse (see (22.19)) matrix for $(b_{jk}(\xi'))$ and

$$f_j = \gamma B_j(D) \cdot R_0 f, \quad \text{where} \quad R_0 f = P_+ Op(r_+/a_+)\theta_+ Op(r_-/a_-)Lf.$$

Using the Paley–Wiener theorem 18.4, one can readily verify that the function $R_0 f$ does not depend on L (compare with P.22.7) and vanishes at $x_n = 0$ together with the derivatives with respect to x_n of order $j < \mu$.

Note that $Au = P_+ Op(a_-)Op(a_+)u_+$, where $u_+ \in H^0(\mathbb{R}^n)$ is the continuation by zero for $x_n < 0$ of the function $u \in H^s(\mathbb{R}^n)$. By virtue of the Paley–Wiener theorem, $\theta_+ Op(r_-/a_-)Op(a_-)f_- = 0$ $\forall f_- \in H^0(\mathbb{R}^n)$, if $P_+ f_- = 0$. Therefore,

$$\begin{aligned} R_0 Au &= P_+ Op(r_+/a_+)\theta_+ Op(r_-/a_-)Op(a_-)Op(a_+)u_+ \\ &= P_+ Op(r_+/a_+)\theta_+ Op(a_+)u_+ + T_1 u \\ &= u + T_2 u, \qquad \text{where} \quad \|T_j u\|_{s+1} \le C\|u\|_s. \end{aligned}$$

The operator R_0 is the regulizer for the operator corresponding to the Dirichlet problem with zero boundary conditions. Similarly, one can prove that, in the case of the half-space, the operator (22.22) is the regulizer for $\mathcal{A}$. $\square$

Estimate (22.21) immediately implies

22.24. Corollary. *If $u \in H^{s-1}(\Omega)$, $\mathcal{A}u \in H^{s,M}(\Omega)$, then $u \in H^s(\Omega)$. In particular, if $u \in H^s(\Omega)$ is the solution of problem* (22.2)–(22.3) *and $f \in C^\infty(\bar{\Omega})$, $g_j \in C^\infty(\Gamma)$, then $u \in C^\infty(\bar{\Omega})$.*

22.25. Proposition. *Under the conditions of Theorem 22.23,* $\operatorname{Ker} \mathcal{A}$*,* $\operatorname{Coker} \mathcal{A}$*, hence, also* $\operatorname{ind} \mathcal{A}$ *do not depend on s.*

Proof. If $u \in H^s$ and $\mathcal{A}u = 0$, then, by virtue of Corollary 22.24, $u \in H^t$ $\forall t > s$, i.e., $\operatorname{Ker} \mathcal{A}$ does not depend on s. Then, since $H^{s,M}$ is the direct sum $\mathcal{A}(H^s)\dot{+}Q$, where Q is a finite dimensional subspace, and since $H^{t,M}$ is dense in $H^{s,M}$ for $t > s$, it follows (see Lemma 2.1 in [**26**]) that $Q \subset H^{t,M}$. Therefore, (accounting Corollary 22.24)

$$H^{t,M} = H^{t,M} \cap H^{s,M} = H^{t,M} \cap \mathcal{A}(H^s)\dot{+}H^{t,M} \cap Q = \mathcal{A}(H^t)\dot{+}Q,$$

i.e., $\operatorname{Coker}\mathcal{A}$ does not depend on s. □

22.26. Remark. $\operatorname{Ker}\mathcal{A}$ and $\operatorname{Coker}\mathcal{A}$ do not depend on s, but when perturbing the operator $\mathcal{A}$ by an operator of lower order[6)] or by an operator with arbitrarily small norm[6)], $\dim\operatorname{Ker}\mathcal{A}$ and $\dim\operatorname{Coker}\mathcal{A}$ can vary. This can bee seen (the reader can easily verify) even in the one-dimensional case. Nevertheless, $\operatorname{ind}\mathcal{A} = \dim\operatorname{Ker}\mathcal{A} - \dim\operatorname{Coker}\mathcal{A}$ does not depend on these perturbations.

[6)] The elliptic theory has been constructed with the help of such operators

Moreover, the following theorem is valid (see, for instance, [**7, 26, 59**]).

22.27. Theorem (on stability of the index). *Let $\Omega \Subset \mathbb{R}^n$, and let the family of elliptic operators*

$$\mathcal{A}_t : H^s(\Omega) \to H^{s,M}(\Omega), \quad \textit{where} \quad t \in [0,1],$$

be continuous with respect to t, i.e.,

$$\|\mathcal{A}_t u - \mathcal{A}_\tau u\|_{s,M} \le C(t,\tau)\|u\|_s, \quad \textit{where} \quad C(t,\tau) \longrightarrow 0 \quad \textit{as} \quad t \longrightarrow \tau.$$

Then $\operatorname{ind}\mathcal{A}_0 = \operatorname{ind}\mathcal{A}_1$.

22.28. Remark. Theorem 22.27 gives us a convenient method for investigation of solvability of elliptic operators $\mathcal{A}u = F$. Actually, assume that, for a family of elliptic operators

$$\mathcal{A}_t = (1-t)\mathcal{A} + t\mathcal{A}_1 : H^s \longrightarrow H^{s,M}$$

it is known that $\operatorname{ind}\mathcal{A}_1 = 0$. Then $\operatorname{ind}\mathcal{A} = 0$. If, moreover, we can establish that $\operatorname{Ker}\mathcal{A} = 0$, then the equation $\mathcal{A}u = F$ is uniquely solvable. If $\dim\operatorname{Ker}\mathcal{A} = 1$, then the equation $\mathcal{A}u = F$ is solvable for any F orthogonal in $H^{s,M}$ to a non-zero function, and the solution is determined uniquely up to the one-dimensional $\operatorname{Ker}\mathcal{A}$.

Let us give (following [**4**]) an example of an elliptic operator of a rather general form, whose index is equal to zero.

22.29. Example. Let $\Omega \Subset \mathbb{R}^n$ be a domain with a smooth boundary Γ and

$$\mathcal{A}_q = \left(a(x,D), b_1(x,D)\big|_\Gamma, \ldots, b_\mu(x,D)\big|_\Gamma\right) : H^s(\Omega) \longrightarrow H^{s,M}(\Omega).$$

Here,

$$a(x,\xi) = \sum_{|\alpha|+k\le 2\mu} a_\alpha(x)\xi^\alpha q^k,\ b_j(x,\xi) = \sum_{|\beta|+l\le m_j} b_{j\beta}(x)\xi^\beta q^l,$$

where $q \geq 0$. Suppose that the *ellipticity with a parameter* holds, i.e.,

$$a_{2\mu}(x,\xi,q) \equiv \sum_{|\alpha|+k=2\mu} a_\alpha(x)\xi^\alpha q^k \neq 0\ \forall(\xi,q)\neq 0\ \forall x\in\bar{\Omega}.$$

Then $a_{2\mu}(x,\eta,q)$ admits (see Lemma 22.16) the factorization

$$a_{2\mu}(x,\eta,q) = a_+(x,\eta,q)a_-(x,\eta,q).$$

Suppose also that the analogue of the Shapiro–Lopatinsky condition 22.18 holds. Namely, for any $x \in \Gamma$, the principal symbols

$$b_j(x,\xi) = \sum_{|\beta|+l\le m_j} b_{j\beta}(x)\xi^\beta q^l\big|_{\xi={}^t\sigma'(x)\eta},\ j=1,\dots,\mu$$

of the boundary operators, considered as polynomials in η_n, are linearly independent modulo the function $a_+(x,\eta,q)$, considered as a polynomial in η_n.

We repeat the proof of Theorem 22.23, preliminarily replacing $\langle\xi\rangle = 1+|\xi|$ in the definition of the norm in the space H^s (see Definition 20.2) by $\langle\xi\rangle = 1+q+|\xi|$. Then, by virtue of obvious inequality

$$\|(1+q+|\xi|)^s\widetilde{u}(\xi)\|_{L^2(\mathbb{R}^n)} \le \frac{1}{q}\|(1+q+|\xi|)^{s+1}\|_{L^2(\mathbb{R}^n)},$$

we obtain that the regulizer R of the operator $\mathcal{A}_q$ (see the proof of Theorem 22.23) satisfies the relations

$$R\cdot\mathcal{A}_q u = u + Tu,\ \|Tu\|_s \le \frac{1}{q}\|u\|_s$$

and

$$\mathcal{A}_q\cdot RF = F + T_1F,\ \|T_1F\|_{s,M} \le \frac{1}{q}\|F\|_{s,M}.$$

Therefore, the operators $1+T$ and $1+T_1$ are for $q \gg 1$ automorphisms of the appropriate spaces, and the equation $\mathcal{A}_q u = F$ for $q \gg 1$ is uniquely solvable. Thus, $\operatorname{ind}\mathcal{A}_q = 0$ for any non-negative q by Theorem 22.27.

The following proposition easily follows from Remark 22.28 and Example 22.29.

22.30. PROPOSITION. *Let $\mathcal{A} : H^s \to H^{s,M}$ be the operator corresponding to the problem from Example 22.21 (for instance, the Dirichlet problem*

$$a(x, D)u \equiv \sum_{|\alpha|=2\mu} a_\alpha(x) D^\alpha u = f \quad \textit{in} \quad \Omega \Subset \mathbb{R}^n,$$

$$\frac{\partial^{j-1} u}{\partial \nu^{j-1}} = g_j \quad \textit{on} \quad \Gamma, \quad j = 1, \dots, \mu$$

for the elliptic operator $a(x, D)$ which satisfies condition (22.14)*) or to the elliptic Poincaré problem (considered in P.22.22). Then* $\operatorname{ind} \mathcal{A} = 0$.

22.31. COROLLARY. *(Compare with P.5.17). The Dirichlet problem*

$$\Delta u = f \in H^{s-2}(\Omega), \quad u = g \in H^{s-1/2}(\Gamma), \quad s \geq 1$$

in a domain $\Omega \Subset \mathbb{R}^n$ with a sufficiently smooth boundary Γ is uniquely solvable. In this case,

$$\|u\|_s \leq C(\|f\|_{s-2} + \|g\|'_{s-1/2}). \tag{22.23}$$

PROOF. By virtue of the maximum principle (Theorem 5.13), $\operatorname{Ker} \mathcal{A} = 0$. Therefore, $\operatorname{Coker} \mathcal{A} = 0$, since $\operatorname{ind} \mathcal{A} = 0$. Furthermore, since $\operatorname{Ker} \mathcal{A} = 0$, the general elliptic estimate (22.21) implies estimate (22.23). Indeed, arguing by contradiction, we take a sequence $\{u_n\}$ such that $\|u_n\|_s = 1$ and $\|\mathcal{A} u_n\|_{s,M} \to 0$. By virtue of the compactness of the embedding $H^s(\Omega)$ in $H^{s-1}(\Omega)$ and estimate (22.21), we can assume that u_n converges in H^s to $u \in H^s$. Since $\|u_n\|_s = 1$, we have $\|u\|_s = 1$. On the other hand, $\|u\|_s = 0$, since $\operatorname{Ker} A = 0$ and $\|\mathcal{A} u\|_{s,M} = \lim \|\mathcal{A} u_n\|_{s,M} = 0$. □

22.32. COROLLARY. *The Neuman problem*

$$\Delta u = f \in H^{s-2}(\Omega), \quad \frac{\partial u}{\partial \nu} = g \in H^{s-3/2}(\Gamma), \quad s > 3/2 \tag{22.24}$$

in a domain $\Omega \Subset \mathbb{R}^n$ with a sufficiently smooth boundary Γ is solvable if and only if

$$\int_\Omega f(x) dx - \int_\Gamma g(\gamma) d\Gamma = 0. \tag{22.25}$$

In this case, the solution u is determined up to a constant.

PROOF. The necessity of (22.25) immediately follows from the Gauss formula (7.5). The first Green formula (or the Giraud–Hopf–Oleinik lemma 5.23) implies that $\operatorname{Ker}\mathcal{A}$ consists of constants. Hence, $\dim\operatorname{Coker}\mathcal{A} = 1$, since $\operatorname{ind}\mathcal{A} = 0$. Therefore, problem (22.24) is solvable, if the right-hand side $F = (f, g)$ satisfies one and only one condition of orthogonality. Thus, the necessary condition (22.25) is also sufficient for solvability of problem (22.24). □

22.33. REMARK. The method for investigation of the solvability of elliptic equations described in Remark 22.28 can be applied in more general situations, for instance, for problems with conditions of conjunction on the surfaces of discontinuity of coefficients [**14**].

22.34. REMARK. The theory of elliptic boundary-value problems for differential operators considered in this section allows a natural generalization onto pseudodifferential operators (see [**19, 66**]). In particular, one can show [**15**] that the equation

$$\epsilon^2 u(x) + \frac{1}{4\pi}\int\limits_{\Omega} \frac{e^{-q|x-y|}}{|x-y|} u(y)\,dy = f(x), \quad \Omega \Subset \mathbb{R}^3$$

for the operator considered in Example 21.3 has, for $\epsilon \geq 0$ and $q \geq 0$, the unique solution $u \in H^{-1}(\Omega)$ for any $f \in C^\infty(\bar{\Omega})$. If $\epsilon = 0$, then

$$u = u_0 + \rho \cdot \delta\big|_\Gamma,\ u_0 \in C^\infty(\bar{\Omega}),\ \rho \in C^\infty(\Gamma),$$

where $\delta\big|_\Gamma$ is the δ-function concentrated on Γ. For $\epsilon > 0$, we have $u \in C^\infty(\bar{\Omega})$ and

$$u(x) = u_0(x) + \frac{1}{\epsilon}\rho(y')\varphi e^{-y_n/\epsilon} + r_0(x, \epsilon),$$

where $\|r_0\|_{L^2} \leq C\sqrt{\epsilon}$, y_n is the distance along the normal form x to $y' \in \Gamma$ and $\varphi \in C^\infty(\bar{\Omega})$, $\varphi \equiv 1$ in a small neighbourhood of Γ (and $\varphi \equiv 0$ outside a slightly greater one).

Addendum
A new approach to the theory of generalized functions (Yu.V. Egorov)

1. *Deficiencies of the distribution theory.* The distribution theory of L. Schwartz was created, mainly, to 1950 and fast has won popularity not only among mathematicians, but also among representatives of other natural sciences. This can be explained to a large degree by the fact that fundamental physical principles can be laid in the basis of this theory; hence its application becomes quite natural. On the other hand, a lot of excellent mathematical results was obtained in last years just due to wide use of the distribution theory. However, it was soon found out that this theory has two essential deficiencies, which seriously hinder its application both in mathematics and in other natural sciences.

The first of them is connected with the fact that in the general case, it is impossible to define the operation of multiplication of distributions so that this operation were associative. It can be seen, for example, from the following reasoning due to L. Schwartz: a product $(\delta(x) \cdot x) \cdot (1/x)$ is defined, since each distribution can be multiplied by an infinitely differentiable function, and is equal to 0. On the other hand, the product $\delta(x) \cdot (x \cdot (1/x))$ is also defined and equal to $\delta(x)$.

Moreover, L. Schwartz has proved the following theorem.

Theorem. *Let A be an associative algebra, in which a derivation operator (i.e., a linear operator $D : A \to A$ such that $D(f \cdot g) = f \cdot D(g) + D(f) \cdot g$) is defined. Suppose that the space $C(\mathbb{R})$ of continuous functions on the real line is a subalgebra in A, and D coincides with* *the usual derivation operator on the set of continuously differentiable functions, and the function, which is identically equal to* 1*, is the unit of the algebra* A*. Then* A *cannot contain an element* δ *such that* $x \cdot \delta(x) = 0$.

Let us show that the product $\delta \cdot \delta$ is not defined in the space of distributions. Let $\omega(x)$ be a function from $C_0^\infty(\mathbb{R})$ such that $\int \omega(x)\,dx = 1$, $\omega(0) = 1$; suppose that $\omega_\varepsilon(x) = \omega(x/\varepsilon)/\varepsilon$. It is natural to assume that $\delta \cdot \delta = \lim \omega_\varepsilon^2$; hence, $(\delta \cdot \delta, \varphi) = \lim \int \omega_\varepsilon^2(x)\varphi(x)\,dx$. However, $(\omega_\varepsilon^2, \omega) = \int \omega_\varepsilon^2(x)\omega(x)\,dx = \varepsilon^{-1} \int \omega^2(x)\omega(\varepsilon x)\,dx \to \infty$ as $\varepsilon \to 0$, that proves our statement. Thus, the distribution theory practically cannot be applied for solution of nonlinear problems.

Another essential deficiency of the distribution theory is connected with the fact that even linear equations with infinitely differentiable coefficients, which are “ideal” for this theory, can have no solutions. For example, this property holds for the equation

$$\partial u/\partial x + ix\partial u/\partial y = f(x, y).$$

It is possible to select an infinitely differentiable function f with a compact support on the plane of variables x, y such that this equation has no solutions in the class of distributions in any neighbourhood of the origin. Actually, such functions f are rather numerous: they form a set of the second category in $C_0^\infty(\mathbb{R}^2)$!

2. *Shock waves.* In gas dynamics, in hydrodynamics, in the elasticity theory and in other areas of mechanics, an important role is played by the theory of discontinuous solutions of differential equations. Such solutions are usually considered when studying the shock waves. By the shock wave we mean a phenomenon, when basic characteristics of a medium have different values on different sides of a surface, which is called the front of the wave. Although, these magnitudes actually vary continuously, their gradient in a neighbourhood of a wavefront is so great that it is convenient to describe them with the help of discontinuous functions.

For example, in gas dynamics a jump of pressure, density and other magnitudes occurs at distances of order 10^{-10} m. The equations of gas dynamics have the form:

$$\rho_t + (\rho v)_x = 0 \qquad \text{(continuity equation)}, \tag{1}$$

$$(\rho v)_t + (\rho v^2 + p)_x = 0 \qquad \text{(equation of motion)}, \qquad (2)$$

$$p = f(\rho, T) \qquad \text{(equation of state)}. \qquad (3)$$

Here ρ is the density of the gas, v is the velocity of particles of the gas, p is the pressure, T is the temperature. The form of the first two equations is divergent that allows one to define generalized solutions with the help of integrations by parts, as it is done in the distribution theory. In this case, it is usually assumed that

$$\rho = \rho_l + \theta(x - vt)(\rho_r - \rho_l), \quad p = p_l + \theta(x - vt)(p_r - p_l),$$

where θ is the Heavyside function, which is equal to 0 for negative values of the argument and equal to 1 for positive values, and smooth functions ρ_l, p_l, ρ_r, p_r are values of the density and pressure, respectively, to the left and to the right from the surface of the wave front.

The essential deficiency of this exposition is the use of only one, common Heavyside function. If we replace it by a smooth function θ_ε, for which the passage from zero value to unit one is carried out on a small segment of length ε, then condition (3) will be broken in this passage area and it can affect the results of calculations!

The analysis of this situation suggests a natural solution: for description of the functions ρ and p we should use different functions θ_ε. In the limit, as $\varepsilon \to 0$, these functions tend to one common Heavyside function, but for $\varepsilon \neq 0$, these functions should be such that condition (3) holds.

Actually, such situation occurs in applied mathematics rather often: for proper, adequate description of a phenomenon with the use of discontinuous functions, it is necessary to remember the way of approximation of these discontinuous functions by smooth ones. It is in this impossibility of such remembering, principal for the distribution theory, the main deficiency of this theory consists, which does not allow one to use it in nonlinear problems.

We are now going to describe a new theory which includes the distribution theory and at the same time is free from the deficiency indicated.

Such a theory was constructed firstly by J.-F. Colombeau (see [**10**] and [**11**]). We give here another version of this theorem which is simpler and more general.

3. *A new definition of generalized functions.* How large a space of generalized functions were, the space of infinitely differentiable functions must be dense in it. This quite natural convention is generally accepted, it is quite justified by practical applications, and we have no reasons to abandon it. Therefore, it is natural to define the space of generalized functions as the completion of the space of infinitely differentiable functions in some topology, which in effect defines the required space. For example, the space of distributions can be defined, by considering all sorts of sequences of infinitely differentiable functions $\{f_j\}$ such that any sequence $\int f_j(x)\varphi(x)\,dx$ has a finite limit as $j \to \infty$, if $\varphi \in C_0^\infty$.

Let Ω be a domain in the space $\mathbb{R}^n$. Consider the space of sequences $\{f_j(x)\}$ of functions infinitely differentiable in Ω. Two sequences $\{f_j(x)\}$ and $\{g_j(x)\}$ from this space are called *equivalent*, if for any compact subset K of Ω there exists $N \in \mathbb{N}$ such that $f_j(x) = g_j(x)$ for $j > N$, $x \in K$. The set of sequences which are equivalent to $\{f_j(x)\}$ is called a *generalized function.* The *space of generalized functions* is designated $\mathcal{G}(\Omega)$.

If a generalized function is such that, for some its representative $\{f_j(x)\}$ and any function φ from $\mathcal{D}(\Omega)$ there exists

$$\lim \int f_j(x)\varphi(x)\,dx,$$

then we can define a distribution corresponding to this generalized function. Conversely, any distribution $g \in \mathcal{D}'(\Omega)$ is associated with a generalized function which is defined by the representative $f_j(x) = g \cdot \chi_j * \omega_\varepsilon$, where $\varepsilon = 1/j$, χ_j is a function from the space $C_0^\infty(\Omega)$, which is equal to 1 at points located at distance $\geq 1/j$ from the boundary of the domain Ω. Thus, $\mathcal{D}'(\Omega) \subset \mathcal{G}(\Omega)$.

If a generalized function is defined by a representative $\{f_j(x)\}$, then its derivative of order α is defined as a generalized function, which is given by the representative $\{D^\alpha f_j(x)\}$. The product of two generalized functions, given by representatives $\{f_j(x)\}$ and $\{g_j(x)\}$ is defined as the generalized function corresponding to the representative $\{f_j(x)g_j(x)\}$.

If F is an arbitrary smooth function of k complex variables, then for any k generalized functions $f_1, \dots, f_k$, the generalized function $F(f_1, \dots, f_k)$ is defined. Moreover, such function can also be defined in the case, when F is a generalized function in $\mathbb{R}^{2k}$. For example, the generalized function "δ ", which is defined by the sequence $\{j \cdot \omega(jx)\}$, where $\omega \in C_0^\infty(\Omega)$, $\int \omega(x)\,dx = 1$, corresponds to the Dirac δ-function. Therefore, the generalized function "δ"("δ"(x)) can be defined as a class containing the sequence $\{j \cdot \omega(j^2(jx))\}$. Let us note that the product $x \cdot$ "δ"$(x) \neq 0$, contrary to the distribution theory. It is essential, if we recall the theorem of L. Schwartz, which has been given above.

The generalized function have a locality property. If Ω_0 is a subdomain of Ω, then for any generalized function f its restriction $f\big|_{\Omega_0} \in \mathcal{G}(\Omega_0)$ is defined. Moreover, the restriction can be defined on any smooth subvariety contained in Ω, and even a value $f(x)$ is defined for any point of Ω. One should only to understand that such restriction is a generalized function on an appropriate subvariety. In particular, the values of generalized functions at a point make sense only as *generalized complex numbers*. These numbers are defined as follows.

Consider the set of all sequences of complex numbers $\{c_j\}$. One can introduce a relation of equivalence in this set such that two sequences are equivalent, if they coincide for all sufficiently large values of j. Obtained classes of equivalent sequences are also called *generalized complex numbers*.

A generalized function f *is equal to 0 in* Ω_0, if there are $N \in \mathbb{N}$ and a representative $\{f_j(x)\}$ such that $f_j(x) = 0$ in Ω_0 for $j > N$. The smallest closed set, outside of which $f = 0$, is called the *support* of f. Note, however, that there are paradoxes from the viewpoint of the distribution theory here: it can, for example, happen that the support of f consists of one point, but the value of f at this point is equal to zero!

If the domain Ω is covered with a finite or countable set of domains Ω_j and generalized functions f_j are defined in each of these domains, respectively, so that $f_i - f_j = 0$ on the intersection of the domains Ω_i and Ω_j, then a unique generalized function f is defined whose restriction on Ω_j coincides with f_j.

4. *Weak equality.* By analogy with the distribution theory, a notion of weak equality can be naturally introduced in the theory of generalized functions. Namely, generalized function f and g are weakly equal, $f \sim g$, if for some their representatives $\{f_j(t)\}$ and

$\{g_j(x)\}$ the following condition holds:

$$\lim_{j\to\infty}\int [f_j - g_j]\varphi(x)\,dx = 0,$$

for any function φ from C_0^∞. In particular, two of generalized complex numbers, which are defined by sequences $\{a_j\}$ and $\{b_j\}$, are weakly equal, $a \sim b$, if $\lim(a_j - b_j) = 0$ as $j \to \infty$. It is clear that for distributions, the weak equality coincides with the usual one. If $f \sim g$, then $D^\alpha f \sim D^\alpha g$ for any α. The weak equality is not "too weak", as the following statement shows.

Theorem. *If $f \in \mathcal{G}(\mathbb{R})$ and $f' \sim 0$, and for some function h from $C_0^\infty(\mathbb{R})$ such that $\int h(x)\,dx = a \neq 0$, there exists a finite limit*

$$\lim \int f_j(x)h(x)\,dx = C,$$

then $f \sim \mathrm{const}$.

Proof. By condition, we have

$$\lim \int f_j(x)\varphi'(x)\,dx = 0$$

for any function φ from $C_0^\infty(\mathbb{R})$. Therefore,

$$\lim \int f_j(x)\cdot\left[\sigma(x) - a^{-1}h(x)\int \sigma(x)\,dx\right]dx = 0$$

for any function σ from $C_0^\infty(\mathbb{R})$, i.e.,

$$\lim \int f_j(x)\sigma(x)\,dx = Ca^{-1}\int \sigma(x)\,dx.$$

□

The proven theorem implies, for example, that systems of ordinary differential equations with constants coefficients have no weak solutions except classical ones.

If f and g are functions continuous in a domain Ω, then their product fg is weakly equal to the product of generalized functions corresponding to the functions f and g.

A more general theorem is also valid: if $F \in C^\infty(\mathbb{R}^{2p})$ and $f_1, \dots, f_p$ are continuous functions, then the continuous function $F(f_1, \dots, f_p)$ is weakly equal to the generalized function $F(g_1, \dots, g_p)$, where g_k is a generalized function which is weak equal to f_k.

Note that the concept of weak equality can generate theorems paradoxical from the viewpoint of classical mathematics: for example, the following system of equations is solvable:

$$y \sim 0, \quad y^2 \sim 1.$$

Its solution is, for example, the generalized function which corresponds to $f(\varepsilon, x) = \sqrt{2} \cdot \sin(x/\varepsilon)$.

Consider now the Cauchy problem:

$$\partial u/\partial t \sim F(t, x, u, \dots, D^\alpha u, \dots), \quad u(0, x) \sim \Phi(x), \quad |\alpha| \leq m.$$

Here, $u = (u_1, \dots, u_N)$ is an unknown vector, F and Φ are given generalized functions. It is possible to show that such problem has (and unique) a weak solution in the class of generalized functions without any assumptions concerning the type of the equations. Namely, we let us take any representatives $\{\Phi_j\}$ and $\{F_j\}$ of the classes Φ and F and consider the Cauchy problem:

$$\begin{aligned} &\partial v/\partial t = F_j(t, x, v(t-\varepsilon, x), \dots, D^\alpha v(t-\varepsilon, x), \dots), \\ &v(t, x) = \Phi_j(x) \quad \text{for} \quad -\varepsilon \leq t \leq 0, \end{aligned}$$

where $\varepsilon = 1/j$. It is clear, that

$$v(t, x) = \Phi_j(x) + \int_0^t F_j(s, x, \Phi_j(x), \dots, D^\alpha \Phi_j(x) \dots) ds$$

for $0 \leq t \leq \varepsilon$. Further, in the same way, one can find $v(t, x)$ for $\varepsilon \leq t \leq 2\varepsilon$ and so on. The obtained function $v(x, t) = v_j(t, x)$ is uniquely defined for $0 \leq t \leq T$ and is also a smooth function. Thus, we have constructed a generalized function, which is called a weak solution. If this generalized function belongs to the class C^m, where m is the maximal order of derivatives of u on the right-hand side of the equation, then it satisfies the equation in the usual sense. If the function F is linear in u and in its derivatives, and depends smoothly on t, so that one can consider solutions of the Cauchy problem in the class of distributions, and if the generalized function obtained is a distribution, then it is also a solution in the sense of the distribution theory.

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