Вариационное исчисление 3 курс 5 сем.

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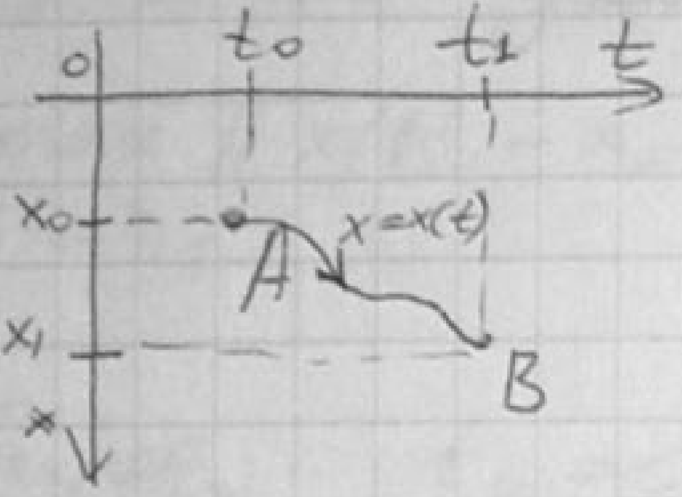
$$\int_a^b f(x, y, \dot y)\, dx$$ 4.09

§1 Введение

П1 1696 И. Бернулли задача о брахистохроне



$A = (t_0, x_0)$

$B = (t_1, x)$

Найти форму желоба, в к-го время движ t мат. точка под действием силы тяжести было бы наименьшим $(t \to \min)$

$x = x(t) \in C^1, \quad t_0 \le t \le t_1$

Нач. условия (краевые усл.):

$x(t_0) = x_0$

$x(t_1) = x_1$

$\exists\ s$ — время $\Rightarrow$ $(t(s), x(s))$ движ. по желобу

$$v(s) = \frac{d}{ds}\left(t(s), x(s)\right) = \left(t'(s), x'(s)\right)$$

$$\frac{m\,|v(s)|^2}{2} = mg\left(x(s) - x_0\right)$$

$$|v(s)| = \sqrt{2g\left(x(s) - x_0\right)}$$

ds: $dl = |v(s)| \cdot ds$

$$dl = \sqrt{2g(x(s) - x_0)}\, ds$$

$$dl = \sqrt{t'^2(s) + x'^2(s)}\, ds$$

$$x'(s) = \frac{dx}{dt} = \frac{dx}{dt} \cdot \frac{dt}{ds} = x'(t) \cdot t'(s)$$

$$\sqrt{1 + x'^2(t)} \cdot t'(s)\, ds = \sqrt{2g(x(s) - x_0)}\, ds$$

$$\Rightarrow ds = \frac{\sqrt{1 + x'^2(t)}}{\sqrt{2g(x(t) - x_0)}}\, dt$$

$$\Rightarrow s(t_1) - s(t_2) = \int_{t_0}^{t_1} \frac{\sqrt{1 + x'^2(t)}}{\sqrt{2g(x(t) - x_0)}}\, dt \longrightarrow \min$$

$$\begin{cases} J(x) = \int_{t_0}^{t_1} \sqrt{\dfrac{1+\dot x^2(t)}{2g\,(x(t)-x_0)}}\,dt \longrightarrow \min \\ x(t_0) = x_0 \quad , \; x(t_1) = x_1 \end{cases}$$

п.2 Постановка задач. вариационного исчисления

$$x(t) = (x_1(t), \dots, x_n(t))^T \quad t\in[t_0,t_1]$$

(1) $J(x) = \int_{t_0}^{t_1} f(t, x(t), \dot x(t))\,dt \longrightarrow \min$

при ограничениях

(2) $\Phi_0(x(t_0)) = 0$, $\Phi_1(x(t_1)) = 0$ (краевые условия)

(3) $\int_{t_0}^{t_1} h_i(t, x(t), \dot x(t))\,dt = H_i$, $i = \overline{1,m}$ ← изопериметрич. ограничение

задан. ф-ции от $n+2$ перем.

(4) $g_j(t, x(t), \dot x(t)) = 0 \quad j = \overline{1,k}$ (связи)

$t_0 < t < t_1$

(1) + (2) ← простейшая задача вариационного исчисления

(1) + (2) + (3) ← изопериметрич. задача — " — — " —

(1) + (2) + (4) — задача вариац. исчисления со связями

Опр 1 Говорим, что $x_*(t) \in C^1[t_0,t_1]$ доставляет слабый лок. минимум, если

а) ф-ция $x_*(t)$ удовл. огранич. задачи;

б) $\exists\, \varepsilon > 0$, для $\forall\, x(t)$, удовл. ограничениям задачи и неравенству $\max\{\max_{[t_0,t_1]} \|x(t) - x_*(t)\|, \max \|\dot x(t) - \dot x_*(t)\|\} < \varepsilon$

справедливо нер-во $J(x) \geqslant J(x_*)$

Опр 2 Говорим, что $x_*(t) \in C^1[t_0,t_1]$ доставл. сильный лок. мин., если

а) ф-ия $x_*(t)$ удовл. огран задачи;

б) $\exists \varepsilon>0$ для $\forall x(t)$, удовл. огран. задачи и нер-ву $\max\limits_{[t_0,t_1]} \|x(t)-x_*(t)\| < \varepsilon$ справедливо нер-во $J(x) \geq J(x_*)$

Пример
$t_0=0$
$t_1=1$
$x_*(t)=0$
$x_\ell(t) = \frac{1}{\ell} \cos \pi \ell^2 t$

$$\max_{[0,1]} |x_\ell(t) - x_*(t)| = \frac{1}{\ell} < \varepsilon$$

$$\max_{[0,1]} |\dot{x}_\ell(t) - \dot{x}_*(t) = \pi\ell \to +\infty \text{ при } \ell \to +\infty$$

кусочно-непр (дифф)

Сильный лок мин — C (непр ф-ии) $\nexists$

Слаб. C^1

Теорема 1 Пусть $x_*(t) \in C^1[t_0,t_1]$ доставляет сильный лок минимум функционала $J(x)$.
Тогда эта ф-ия доставляет и слабый лок. мин.

Лемма (о скруглении углов)

Пусть в простейшей задаче вариационного исчисления (ВИ) непр по всем своим аргументам $(f(t,x,\dot{x}) \in C)$

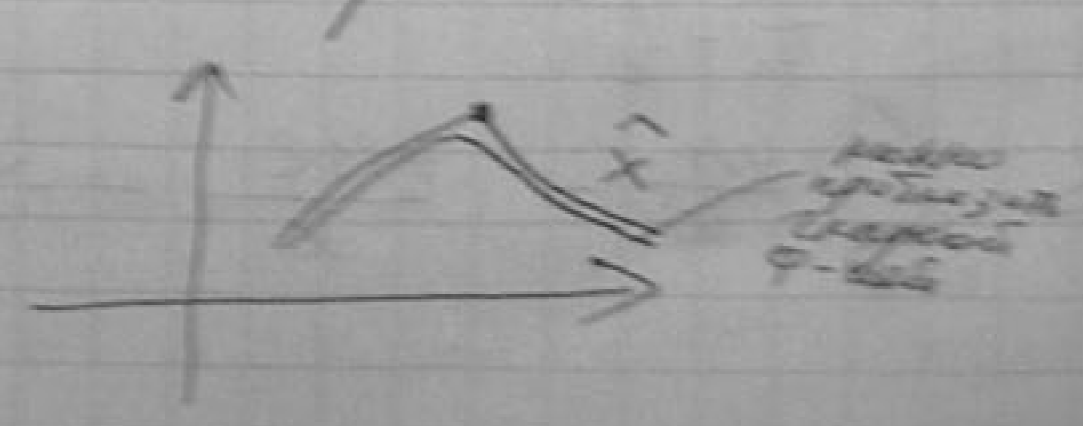
Пусть $\hat{x}(t)$ — кусочно-дифф. на $[t_0,t_1]$

Тогда $\exists \{x_k(t)\}_{k\geq 1}$, $x_k(t) \in C^1[t_0,t_1]$:

1) $x_k(t_0) = \hat{x}_k(t_0)$, $x_k(t_1) = \hat{x}(t_1)$

2) $\max \|x_k(t) - \hat{x}(t)\| \to 0$ при $k \to +\infty$

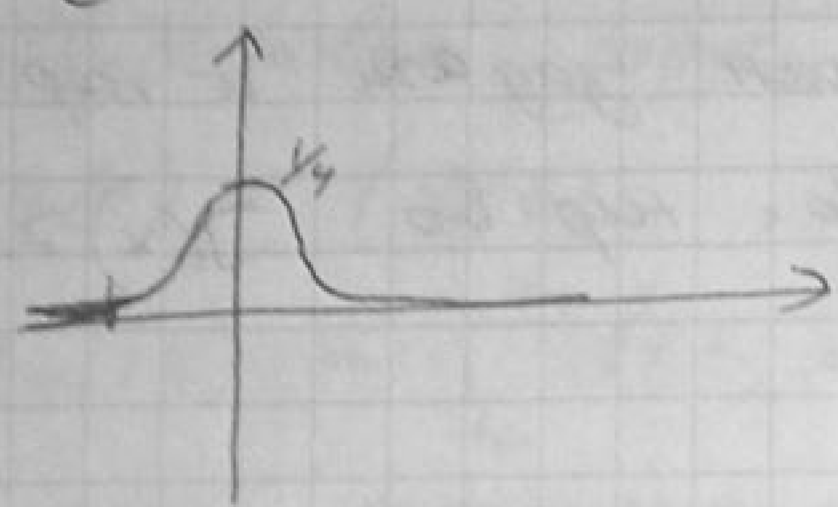
3) $\lim\limits_{k\to+\infty} J(x_k) = J(\hat{x})$



□ Док-во: $n=1$

$$a(t) = \begin{cases} \frac{1}{4}(1-|t|)^2 & |t| \le 1 \\ 0 & |t| > 1 \end{cases}$$

$a(t) \in C$



$$|a(t)| \le \frac{1}{4}$$

$$|\dot{a}(t)| \le \frac{1}{2}$$

Пусть τ_i $i=\overline{1,l}$ — точки разрыва $\hat{x}(t)$

$$\Delta_i = \dot{\hat{x}}(\tau_i+0) - \hat{x}(\tau_i - 0) \text{ — скачок в точке}$$

Рассм. $h_k(\tau_i, t) = \frac{1}{k} a(k(t-\tau_i))$

$$|h_k(\tau_i, t)| \le \frac{1}{4k}$$

$$|\dot{h}_k(\tau_i, t)| \le \frac{1}{2}$$

$$\Rightarrow x_k(t) = \hat{x}(t) + \sum_{i=1}^{l} h_k(\tau_i, t)\Delta_i$$

По построению $x_k(t) = \hat{x}(t)$ вне $[\tau_i - \frac{1}{k}, \tau_i + \frac{1}{k}]$ $i = \overline{1,l}$

Пусть $\Delta = \max\limits_{1 \le i \le l} |\Delta_i|$

$$|x_k(t) - \hat{x}(t)| = \left|\sum_{i=1}^{l} h_k(\tau_i, t)\Delta_i\right| \le \frac{l\Delta}{4k} \text{ при } k \to \infty$$

$$|\dot{x}_k(t) - \dot{\hat{x}}(t)| = \left|\sum_{i=1}^{l} \dot{h}_k(\tau_i, t)\Delta_i\right| \le \frac{l\Delta}{2}$$

Рассм. мн-во $\{(t, x, \dot{x}) : t \in [t_0, t_1], |x - \hat{x}(t)| \le \frac{l\Delta}{4k_0},$
$|\dot{x} - \dot{\hat{x}}(t)| \le \frac{l\Delta}{2}\}$

$$|f(t, x, \dot{x})| \le M$$

$$\Rightarrow |J(x_k) - J(\hat{x})| = \left|\int_{t_0}^{t_1} f(t, x_k(t), \dot{x}_k(t))dt - \int_{t_0}^{t_1} f(t, \hat{x}(t), \dot{\hat{x}}(t))dt\right| =$$

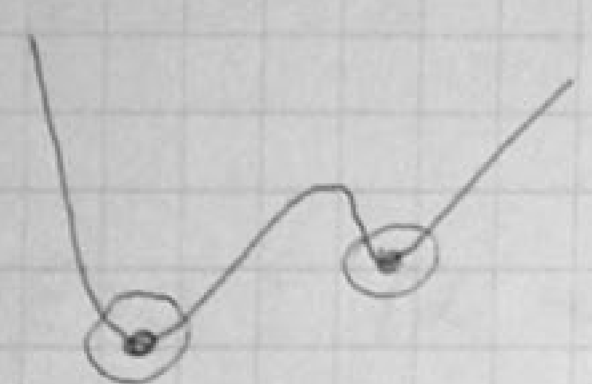
$$= \left|\sum_{i=1}^{n} \int_{\tau_i - \frac{1}{k}}^{\tau_i + \frac{1}{k}} (f(t, x_k(t), \dot{x}_k(t)) - f(t, \hat{x}(t), \dot{\hat{x}}(t)))dt\right| \le$$

$$\le \sum_{i=1}^{l} \frac{4M}{k} = \frac{4Ml}{k} \longrightarrow 0 \text{ при } k \to +\infty$$

11.09.08

Теорема 2

В простой задаче ВИ абсолютно слабый лок min совпадает с абсолютно сильным min



II Необх условие

Пусть $x_*(t)$ доставляет слабый или сильный лок. min

Рассм. $x_\alpha(t)$ а) $x_0(t) = x_*(t)$

б) при малых α $x_\alpha(t)$ удовлетворяет условиям задачи

Если $x_*(t)$ доставляет слабый лок. min и при малых α $\max\{\max_{t\in[t_0,t_1]} \|x_\alpha(t) - x_*(t)\|, \max \|\dot{x}_\alpha(t) - \dot{x}_*(t)\|\} < \varepsilon$

$I(\alpha)$ имеет в $\alpha = 0$ лок. min

C^1

$\dot{I}(0) = 0 \leftarrow$ первая вариация функционала

Теорема 3 Если ф-ия $x_*(t)$, $t\in[t_0,t_1]$ доставляет слабый лок min, то первая вариация $=0$ ($\dot{I}(0)=0$)

Если $x_*(t)$ доставляет сильный лок min и при малых α $\max \|x_\alpha(t) - x_*(t)\| < \varepsilon$, $I(\alpha)$ имеет в $\alpha=0$ лок min

$\dot{I}(+0) \geq 0$ (производная справа)

II. Задача оптимального управления (ЗОУ)

$$\begin{cases} J(u) = \int_{t_0}^{t_1} f_0(t, y(t), u(t))\,dt \longrightarrow \min \\ \dot{y}(t) = f(t, y(t), u(t)) \\ \Phi_0(y(t_0)) = 0, \quad \Phi_1(y(t_1)) = 0 \\ u(t) \in U \end{cases}$$

$y \in R^k$, $u \in R^l$

$$x = (\bar{x}, \bar{\bar{x}}) = (\underbrace{x_1, \ldots, x_k}_{\bar{x}}, \underbrace{x_{k+1}, \ldots x_{k+l}}_{\bar{\bar{x}}})$$

$$\bar{\bar{x}}(t) = \int_{t_0}^{t} u(s)\,ds$$

$\Downarrow$

$\dot{\bar{\bar{x}}}(t) = u(t) \qquad \bar{x}(t) = y(t)$

$\bar{\bar{x}}(t_0) = 0$

$$J(x) = \int_{t_0}^{t_1} f_0(t, \bar{x}(t), \dot{\bar{\bar{x}}}(t))\,dt \xrightarrow{} \min$$

$\dot{\bar{x}}(t) - f(t, \bar{x}(t), \dot{\bar{\bar{x}}}(t)) = 0$

$\Phi_0(\bar{x}(t_0)) = 0 \qquad \Phi_1(\bar{x}(t_1)) = 0$

$\bar{\bar{x}}(t_0) = 0$

$\dot{\bar{\bar{x}}}(t) \in U$

~~КЗ~~ ЛКЗОУ $\leftarrow$ лин. задачи ОУ (ЛКЗОУ)

§1 Простейшая задача ВИ. Уравнение Эйлера.

Рассм. задачу

$$\begin{cases} J(x) = \int_{t_0}^{t_1} f(t, x(t), \dot{x}(t))\,dt \to \min \\ x(t_0) = x_0 \\ x(t_1) = x_1 \end{cases}$$

$f(\cdot, \cdot, \cdot) \in C^2$

<u>Теорема 4</u> $\exists\, x_*(t) \in C^1[t_0, t_1]$ доставляет слабый лок min. Тогда $\frac{\partial f}{\partial x}(t, x_*(t), \dot{x}_*(t)) - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}}(t, x_*(t), \dot{x}_*(t)) = 0$

<u>Замечание</u> $\exists\, x_*(t) \in C^2[t_0, t_1]$

$$\frac{d}{dt}\frac{\partial f}{\partial \dot{x}}(t, x_*(t), \dot{x}_*(t)) = \frac{\partial^2 f}{\partial t \partial \dot{x}}(t, x_*(t), \dot{x}_*(t)) + \frac{\partial^2 f}{\partial x \partial \dot{x}}(t, x_*(t), \dot{x}_*(t))\dot{x}_*(t) + \frac{\partial^2 f}{\partial \dot{x}^2}(t, x_*(t), \dot{x}_*(t))\ddot{x}_*(t)$$

Опр Решение ур-ния Эйлера — экстремали

Док-во: Рассм. $\forall \eta(t) \in C^1[t_0, t_1]$

$$\eta(t_0) = \eta(t_1) = 0$$

$$x_\alpha(t) = x_*(t) + \alpha\eta(t)$$

$$\dot{x}_\alpha(t) = \dot{x}_*(t) + \alpha\dot{\eta}(t)$$

$$\forall \varepsilon > 0 : |\alpha| < \frac{\varepsilon}{\max\{\max\limits_{[t_0,t_1]}\|\eta(t)\|, \max\limits_{[t_0,t_1]}\|\dot{\eta}(t)\|\}}$$

$$I(\alpha) = J(x_\alpha) \int_{t_0}^{t_1} f(t, x_\alpha(t), \dot{x}_\alpha(t))\,dt \Rightarrow$$

$\dot{I}(0) = 0 \leftarrow$ необх. условие лок. min

$$\frac{dx_\alpha}{d\alpha}(t) = \eta(t), \quad \frac{d\dot{x}_\alpha}{d\alpha}(t) = \dot{\eta}(t)$$

$$\Rightarrow \quad 0 = \int_{t_0}^{t_1} \Big[\Big(\frac{\partial f}{\partial x}(t, x_*(t), \dot{x}_*(t)), \eta(t)\Big) + \Big(\frac{\partial f}{\partial \dot{x}}(t, x_*(t), \dot{x}_*(t)), \dot{\eta}(t)\Big)\Big] dt$$

$$\frac{\partial f}{\partial x}(t, x_*(t), \dot{x}_*(t)) = \frac{d}{dt}\Big(\int_{t_0}^{t} \frac{\partial f}{\partial x}(s, x_*(s), \dot{x}_*(s))\,ds + C\Big)$$

$$\int_{t_0}^{t_1}\Big(\frac{\partial f}{\partial x}(t, x_*(t), \dot{x}_*(t)), \eta(t)\Big)dt = \Big(\int_{t_0}^{t_1} \frac{\partial f}{\partial x}(s, x_*(s), \dot{x}_*(s))\,ds + C, \eta(t)\Big)\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1}\Big(\int_{t_0}^{t} \frac{\partial f}{\partial x}(s, x_*(s), \dot{x}_*(s))\,ds + C, \dot{\eta}(t)\Big)dt$$

$$\Rightarrow \int_{t_0}^{t_1}\Big(\underbrace{-\int_{t_0}^{t} \frac{\partial f}{\partial x}(s, x_*(s), \dot{x}_*(s))\,ds - C + \frac{\partial f}{\partial \dot{x}}(t, x_*(t), \dot{x}_*(t))}_{\dot{\eta}(t)\ (*)}, \dot{\eta}(t)\Big)dt = 0$$

$$C: \int_{t_0}^{t_1}\Big[-\int_{t_0}^{t} \frac{\partial f}{\partial x}(s, x_*(s), \dot{x}_*(s))\,ds - C + \frac{\partial f}{\partial \dot{x}}(t, x_*(t), \dot{x}_*(t))\Big]dt = 0$$

$$C = \frac{1}{t_1 - t_0}\int_{t_0}^{t_1}\Big(-\int_{t_0}^{t} \frac{\partial f}{\partial x}(s, x_*(s), \dot{x}_*(s))\,ds + \frac{\partial f}{\partial \dot{x}}(t, x_*(t), \dot{x}_*(t))\Big)dt$$

$$\eta(t) = \int_{t_0}^{t}\Big(-\int_{t_0}^{s} \frac{\partial f}{\partial x}(\tau, x_*(\tau), \dot{x}_*(\tau))\,d\tau - C + \frac{\partial f}{\partial \dot{x}}(s, x_*(s), \dot{x}_*(s))\Big)ds$$

$$\dot{\eta}(t) = (*)$$

$$\int_{t_0}^{t_1} \underbrace{(\dot\eta(t), \dot\eta(t))}_{\forall}\, dt = 0 \quad \Rightarrow \dot\eta(t) = 0$$

$$-\int_{t_0}^{t} \frac{\partial f}{\partial x}(s, x_*(s), \dot x_*(s))\, ds - C + \frac{\partial f}{\partial \dot x}(t, x_*(t), \dot x_*(t)) = 0$$

Интегральное ур-ние Эйлера

дифф.-ем

$$\Rightarrow \frac{\partial f}{\partial x}(t, x_*(t), \dot x_*(t)) - \frac{d}{dt}\frac{\partial f}{\partial \dot x}(t, x_*(t), \dot x_*(t)) = 0$$

Случай 1 $\quad f(t, x, \dot x) = f(x, \dot x)$

$$H(x, \dot x) = \left(\dot x, \frac{\partial f}{\partial \dot x}(x, \dot x)\right) - f(x, \dot x)$$

интеграл энергии

$x_*(t)$ – решение ур-ния Эйлера

$$\frac{d}{dt} H(x_*(t), \dot x_*(t)) = \left(\ddot x(t), \frac{\partial f}{\partial \dot x}(x_*(t), \dot x_*(t))\right) + \left(\dot x_*(t), \frac{d}{dt}\frac{\partial f}{\partial \dot x}(x_*(t), \dot x_*(t))\right)$$

$$- \left(\dot x_*(t), \frac{\partial f}{\partial x}(x_*(t), \dot x_*(t))\right) - \left(\ddot x_*(t), \frac{\partial f}{\partial \dot x}(x_*(t), \dot x_*(t))\right) =$$

$$= \left(\dot x_*(t), \frac{d}{dt}\frac{\partial f}{\partial \dot x}(x_*(t), \dot x_*(t)) - \frac{\partial f}{\partial x}(x_*(t), \dot x_*(t))\right) = 0$$

Случай 2 $\quad f(t, x, \dot x) = f(t, \dot x) \Rightarrow$

$$\Rightarrow \frac{\partial f}{\partial \dot x}(x_*(t), \dot x_*(t)) = const$$

интеграл импульса

Случай 3 $\quad f(t, x, \dot x) = f(t, x)$

$\Rightarrow$ ур-ние Эйлера принимает вид

$$\frac{\partial f}{\partial x}(t, x_*(t)) = 0$$

интеграл силы

Пример (Задача о брахистохроне)

$$f(t, x, \dot x) = \frac{1}{\sqrt{2g}} \frac{\sqrt{1+\dot x^2}}{\sqrt{x - x_0}}$$

экв. запись ур-ния Эйлера

$$H(x, \dot x) = \frac{1}{\sqrt{2g}} \left\{ \frac{\dot x^2}{\sqrt{1+\dot x^2}\sqrt{x-x_0}} - \frac{\sqrt{1+\dot x^2}}{\sqrt{x-x_0}} \right\} = C_0$$

$$\Rightarrow \sqrt{1+\dot x^2}\cdot\sqrt{x-x_0} = -\frac{1}{\sqrt{2g}\, C_0} \Rightarrow 1 + \dot x^2 = \frac{c}{x - x_0}$$

$$\dot{x}(t) = \pm\sqrt{\frac{c}{x(t)-x_0} - 1}$$

$x_0 < x_1 \Rightarrow$ „—" не подходит $\Rightarrow$ рассмотрим только с „+"

Введем параметр

$\Rightarrow$

$\dot{x} = \operatorname{ctg} s$

$x - x_0 = C\sin^2 s \Rightarrow \boxed{x = x_0 + \frac{c}{2} - \frac{c}{2}\cos 2s}$

$dx = 2C\sin s\cos s\, ds = C\sin 2s\, ds$ в зависимости от пар-ра s меняется x

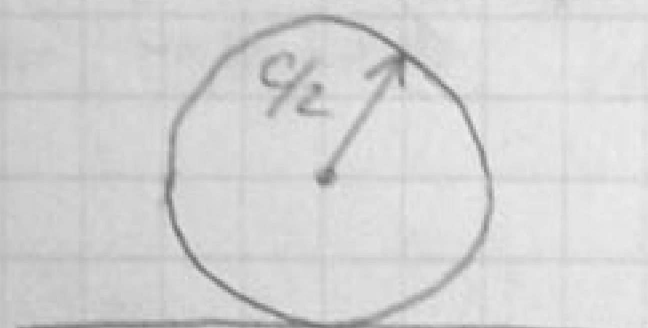
$\frac{dx}{dt} = \operatorname{ctg} s$

$$\underline{dt} = \frac{dx}{\operatorname{ctg}} = \frac{C\sin 2s\, ds}{\operatorname{ctg} s} = \underline{2C\sin^2 s\, ds} = C(1-\cos 2s)\, ds$$

$\Rightarrow \Rightarrow \boxed{t = Cs - \frac{c}{2}\sin 2s + d}$ циклоида

как меняется t в зав-ти от пар-ра s

лишь необходимое условие!



Пример 2 (гармонический осциллятор)

$$\begin{cases} J(x) = \int_0^{3\pi/2} (\dot{x}^2 - x^2)\, dt \to \min \\ x(0) = 0, \quad x\left(\frac{3\pi}{2}\right) = 0 \end{cases}$$

$\frac{\partial f}{\partial \dot{x}} = 2\dot{x}, \quad \frac{\partial f}{\partial x} = -2x$

$\ddot{x} + x = 0 \qquad x(t) = C_1\cos t + C_2\sin t$

$\Rightarrow$ экстремаль $\boxed{x_*(t) = 0}$

решение ур-ние Эйлера

$x_k(t) = \frac{1}{k}\sin\frac{2t}{3}$

$$\max_{[0,\frac{3\pi}{2}]} |x_k(t) - x_*(t)| = \frac{1}{k}\max_{[0,\frac{3\pi}{2}]}\left|\sin\frac{2t}{3}\right| = \frac{1}{k} \xrightarrow[k\to\infty]{} 0$$

$$\max_{[0,\frac{3\pi}{2}]} |\dot{x}_k(t) - \dot{x}_*(t)| = \frac{2}{3k}\max_{[0,\frac{3\pi}{2}]}\left|\cos\frac{2t}{3}\right| = \frac{2}{3k} \xrightarrow[k\to\infty]{} 0$$

$$J(x_k) = \frac{1}{k^2}\int_0^{3\pi/2}\left(\frac{4}{9}\cos^2\frac{2t}{3} - \sin^2\frac{2t}{3}\right) dt = \frac{1}{k^2}\cdot\frac{3\pi}{4}\left(\frac{4}{9} - 1\right) =$$

$$= -\frac{1}{k^2}\cdot\frac{3\pi}{4}\cdot\frac{5}{9} < 0 = J(x_*)$$

Пример 3 (пример Гильберта)

$$\begin{cases} J(x) = \int_0^1 t^{2/3} \dot{x}^2 dt \to \min \\ x(0) = 0, \ x(1) = 1 \end{cases}$$

$$\frac{d}{dt}(t^{2/3} \cdot 2\dot{x}) = 0 \Rightarrow \dot{x}(t) = c \cdot t^{-2/3}$$

$$x(t) = C_1 t^{1/3} + C_2$$

$$\Rightarrow x_*(t) = t^{1/3} \leftarrow \text{экстремаль} \notin C^1[0,1]$$

Покажем, что она доставляет глобальный min среди кусочно-дифф ф-ций $x(t)$, удовлетв. краевым условиям и для которых $J(x) < \infty$

$$\forall h(t) = x(t) - x_*(t), \ t \in [0,1]$$

$$J(x) - J(x_*) = J(x_* + h) - J(x_*) = \int_0^1 t^{2/3}(\dot{x}_* + \dot{h})^2 dt - \int_0^1 t^{2/3} \dot{x}_*^2 dt =$$

$$= \int_0^1 t^{2/3}(2\dot{x}_*(t)\dot{h}(t) + \dot{h}^2(t))dt =$$

$$= \cancel{\int_0^1 \frac{2}{3} \dot{h}(t) dt}^{\,0} + \int_0^1 t^{2/3} \dot{h}^2(t) dt \geq 0$$

Пример 4 (Пример Вейерштрасса)

$$\begin{cases} J(x) = \int_0^1 t^2 \dot{x}^2 dt \to \min \\ x(0) = 0, \ x(1) = 1 \end{cases}$$

Ур-ние Эйлера $\frac{d}{dt}(t^2 \cdot 2\dot{x}) = 0 \Rightarrow \dot{x} = \frac{c}{t^2} \Rightarrow$

$$x(t) = \frac{C_1}{t} + C_2$$

$x(t)$ не удовл. ~~к~~ краевым усл.

$\Rightarrow$ В этой задаче экстремалей нет

$$J(x) \geq 0$$

Для $\forall$ кусочно дифф ф-ции $x(t) \neq 0$ $J(x) > 0$

$$\inf J(x) = 0$$

$$x_n(t) = \frac{\operatorname{arctg} nt}{\operatorname{arctg} n}$$

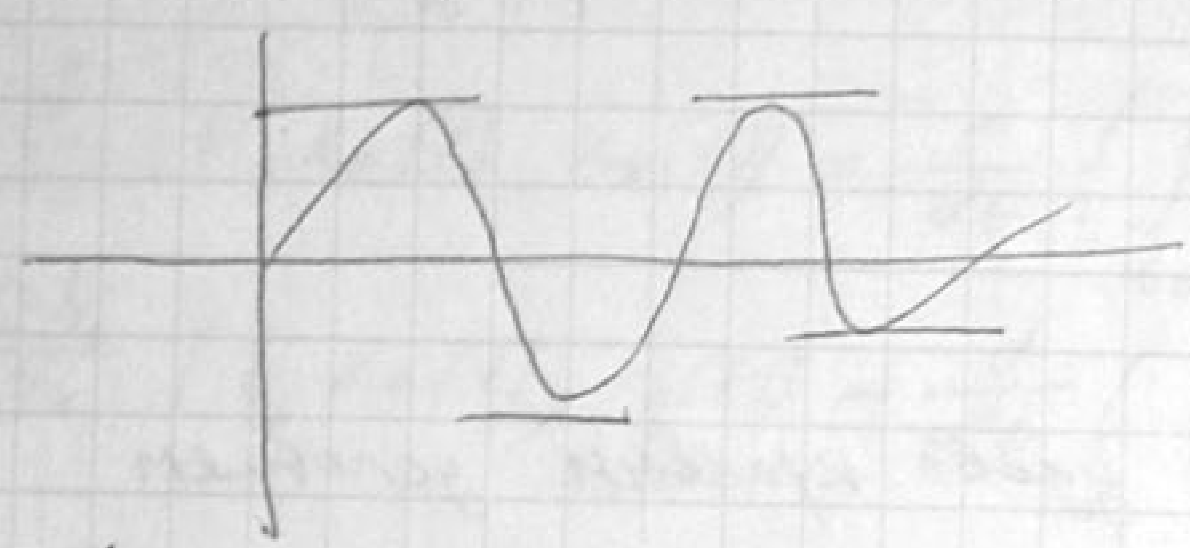
$$J(x_n) = \int_0^1 t^2 \frac{n^2}{(1+n^2t^2)\operatorname{arctg}^2 n} dt \leq \int_0^{1/n} \frac{dt}{\operatorname{arctg}^2 n} + \int_{1/n}^1 \frac{dt}{n^2t^2 \operatorname{arctg}^2 n} \xrightarrow[n\to\infty]{} 0$$

Пример 5 (Больц)

$$\begin{cases} J(x) = \int\limits_0^1 \{(1-\dot{x}^2)^2 + x^2\} dt \longrightarrow \min \\ x(0)=0 \\ x(1)=0 \end{cases}$$

Покажем $\inf J(x) = 0$

$$x_n(t) = \int\limits_0^t \operatorname{sgn}(\sin 2\pi n \tau)\, d\tau, \quad n \geq 1$$



$\{x_n(t)\} \rightrightarrows 0 \qquad |\dot{x}(t)| = 1$ почти всюду

$J(x_n) \longrightarrow 0$

если $x(t) = 0 \Rightarrow J(x) > 0$

$$J(x) = \int\limits_0^1 x^2 dt$$

$x(t) \neq 0 \Rightarrow J(x) = 1$

Точная нижняя грань $= 0$, но не достигает на $\forall$ мн-ве

§2 Задача ВИ с изопериметрическими ограничениями

$$(*) \begin{cases} J(x) = \int\limits_{t_0}^{t_1} f(t, x(t), \dot{x}(t))\, dt \longrightarrow \min \\ x(t_0) = x_0, \; x(t_1) = x_1 \\ \int\limits_{t_0}^{t_1} h_i(t, x(t), \dot{x}(t))\, dt = H_i \quad i = \overline{1,m} \end{cases}$$

$f(t, x, \dot{x}),\ h_i(t, x, \dot{x}) \in C^1_{t x \dot{x}}$

$f(t, x, \dot{x}),\ h_i(t, x, \dot{x}) \in C^2_{\dot{x}}$

Теорема) $\exists\, x_*(t) \in C^1[t_0, t_1]$, доставл. слаб. лок. min в $(*)$. Тогда $\exists\, (\lambda_0, \lambda_1, \dots, \lambda_m) \neq 0$:

$$L(t, x, \dot{x}, \lambda) = \lambda_0 f(t, x, \dot{x}) + \sum_{i=1}^m \lambda_i h_i(t, x, \dot{x})$$

$$\Rightarrow \frac{\partial L}{\partial x}(t, x_*(t), \dot{x}_*(t), \lambda) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(t, x_*(t), \dot{x}_*(t), \lambda) = 0$$

Док-во $z_0(t) = \frac{\partial f}{\partial \dot{x}}(t, x_*(t), \dot{x}_*(t)) - \int_{t_0}^{t} \frac{\partial f}{\partial x}(s, x_*(s), \dot{x}_*(s))ds + c_0$

$$z_i(t) = \frac{\partial h_i}{\partial \dot{x}_*}(t, x_*(t), \dot{x}_*(t)) - \int_{t_0}^{t} \frac{\partial h_i}{\partial x}(s, x_*(s), \dot{x}_*(s))ds + c_i$$

$i = \overline{1,m}$

Выбирем константы c_i $i = \overline{0,m}$; $\int_{t_0}^{t_1} z_i(s)ds = 0$ $i = \overline{0,m}$

$\eta_i(t) = \int_{t_0}^{t} z_i(s)\,ds$, $i = \overline{0,m}$

$\Downarrow$

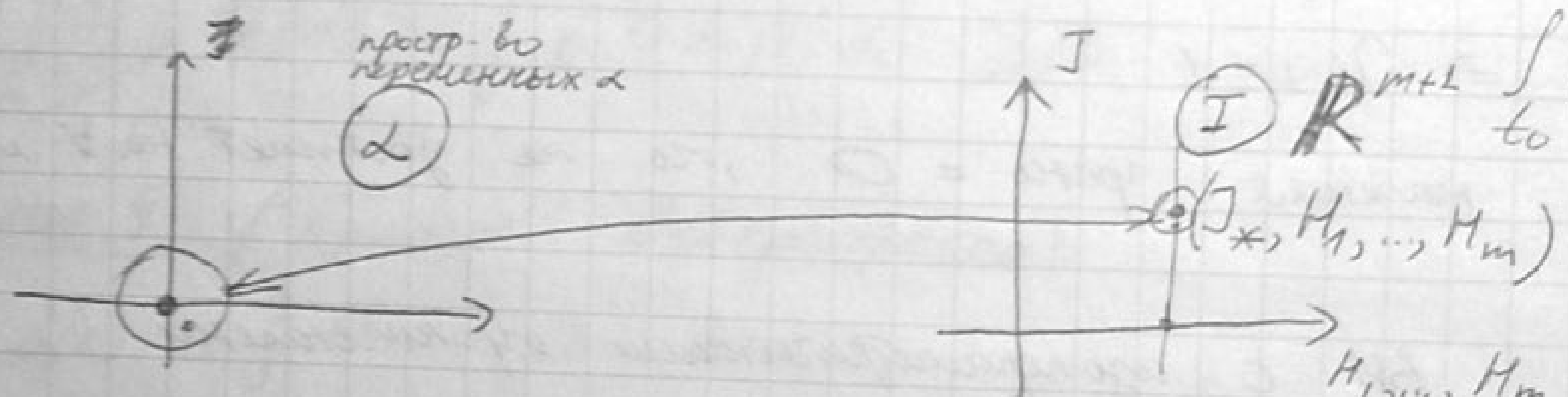
$\eta_i(t_0) = \eta_i(t_1) = 0$ $i = \overline{0,m}$

$x_\alpha(t) = x_*(t) + \sum_{i=0}^{m} \alpha_i \eta_i(t)$ удовл. краевым условием

$\dot{x}_\alpha(t) = \dot{x}_*(t) + \sum_{i=0}^{m} \alpha_i \dot{\eta}_i(t)$

$$I(\alpha) = (I_0(\alpha_0, \alpha_1, \dots, \alpha_m), I_1(\alpha_0, \alpha_1, \dots, \alpha_m), \dots, I_m(\alpha_0, \alpha_1, \dots \alpha_m))$$

$I_0 = \int_{t_0}^{t_1} f(t, x_\alpha(t), \dot{x}_\alpha(t))dt$, $I_1 = \int_{t_0}^{t_1} h_1(t, x_\alpha(t), \dot{x}_\alpha(t))dt$, $I_m = \int_{t_0}^{t_1} h_m(t, x_\alpha(t), \dot{x}_\alpha(t))dt$



При $\alpha = 0$ $\left. \frac{D(I_0, I_1, \dots I_m)}{D(\alpha_0, \alpha_1, \dots, \alpha_m)} \right|_{\alpha=0} = 0$

<u>от противного</u>

Предположим $\exists$ $\left. \frac{D(I_0, I_1, \dots, I_m)}{D(\alpha_0, \alpha_1, \dots, \alpha_m)} \right|_{\alpha=0} \neq 0$

Возьмем точку из окрестности в $\mathbb{R}^{m+1}$, ей будет соотв. точка в α $\Rightarrow$ противоречие

$$\frac{\partial x_\alpha}{\partial \alpha_i}(t) = \eta_i(t)$$
$$\frac{\partial \dot x_\alpha}{\partial \alpha_i}(t) = \dot\eta_i(t)$$
$$\frac{\partial I_0}{\partial \alpha_j}(0) = \int_{t_0}^{t_1} \left(\left(\frac{\partial f}{\partial x}(t, x_*(t), \dot x_*(t)), \eta_j(t)\right) + \left(\frac{\partial f}{\partial \dot x}(t, x_*(t), \dot x_*(t)), \dot\eta_j(t)\right)\right) dt \overset{1}{=}$$
$$\frac{\partial I_i}{\partial \alpha_j}(0) = \int_{t_0}^{t_1} \left\{ \left(\frac{\partial h}{\partial x}(t, x_*(t), \dot x_*(t)), \eta_j(t)\right) + \left(\frac{\partial h_i}{\partial \dot x}(t, x_*(t), \dot x_*(t)), \dot\eta_j(t)\right)\right\} dt \overset{2}{=} \quad i=\overline{1,m}$$
$$\frac{\partial f}{\partial x}(t, x_*(t), \dot x_*(t)) = \frac{d}{dt}\left(\int_{t_0}^{t} \frac{\partial f}{\partial x}(s, x_*(s), \dot x_*(s))\, ds - C_0\right)$$
$$\frac{\partial h_i}{\partial x}(t, x_*(t), \dot x_*(t)) - \frac{d}{dt}\left(\int_{t_0}^{t} \frac{\partial h_i}{\partial x}(s, x_*(s), \dot x_*(s))\, ds - C_i\right)$$
$$i=\overline{1,m}$$

$\overset{1}{=}$ интегрируя по частям с учетом

$$= \int_{t_0}^{t_1} \left(-\int_{t_0}^{t} \frac{\partial f}{\partial x}(s, x_*(s), \dot x_*(s))\, ds + C_0 + \frac{\partial f}{\partial \dot x}(t, x_*(t), \dot x_*(t)), \dot\eta_j(t)\right) dt$$

$$\overset{2}{=} \; = \int_{t_0}^{t_1} \left(-\int_{t_0}^{t} \frac{\partial h_i}{\partial x}(s, x_*(s), \dot x_*(s))\, ds + C_i + \frac{\partial h_i}{\partial \dot x}(t, x_*(t), \dot x_*(t)), \dot\eta_j(t)\right) dt$$

$$\frac{\partial I_0}{\partial \alpha_j}(0) = \int_{t_0}^{t_1} (z_0(t), z_j(t))\, dt, \quad j=\overline{0,m}$$
$$\frac{\partial I_i}{\partial \alpha_j}(0) = \int_{t_0}^{t_1} (z_i(t), z_j(t))\, dt \quad i=\overline{1,m}$$

$$0 = \begin{vmatrix} \int_{t_0}^{t_1} (z_0(t), z_0(t))\, dt & \dots & \int_{t_0}^{t_1} (z_0(t), z_m(t))\, dt \\ \int_{t_0}^{t_1} (z_1(t), z_0(t))\, dt & \dots & \int_{t_0}^{t_1} (z_1(t), z_m(t))\, dt \\ \dots & \dots & \dots \\ \int_{t_0}^{t_1} (z_m(t), z_0(t))\, dt & \dots & \int_{t_0}^{t_1} (z_m(t), z_m(t))\, dt \end{vmatrix}$$ Якобиан

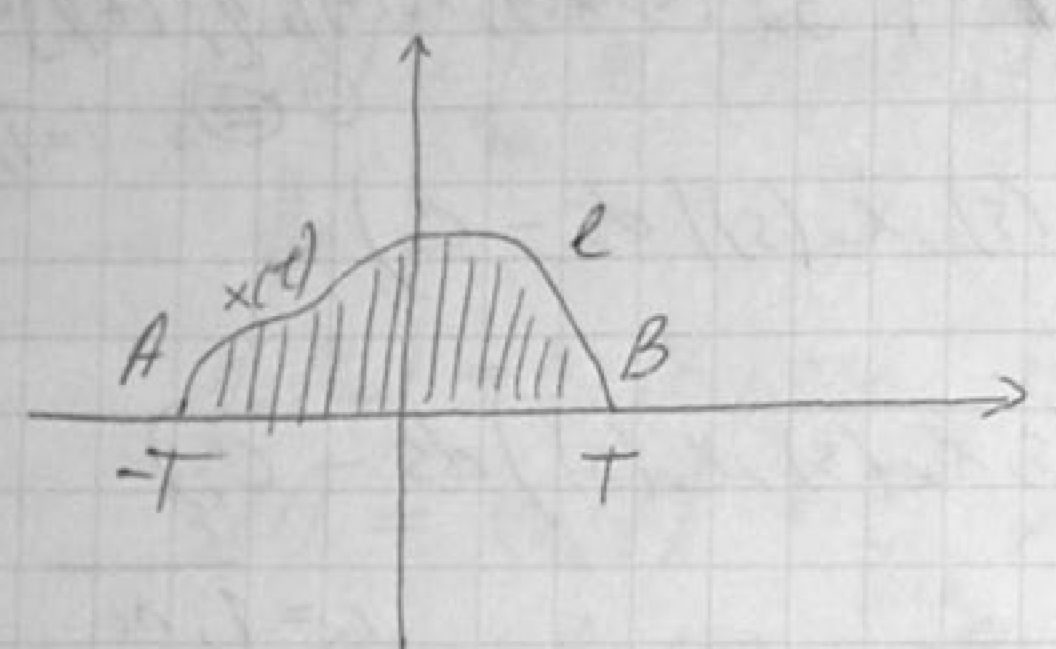
$$\exists\ \lambda_0, \lambda_1, \dots, \lambda_m \neq 0 : \sum_{j=0}^{m} \lambda_j \int_{t_0}^{t_1} (z_i(t), z_j(t))\, dt = 0 \quad i=\overline{0,m} \,|\cdot \lambda_i$$
$$\to \sum_{i=0}^{m}\sum_{j=0}^{m} \lambda_j \lambda_i \int_{t_0}^{t_1} (z_i(t), z_j(t))\, dt = 0$$
$$\int_{t_0}^{t_1} \left(\sum_{i=0}^{m} \lambda_i z_i(t), \sum_{j=0}^{m} \lambda_j z_j(t)\right) dt = 0$$

$$\sum_{i=0}^{m} \lambda_i z_i(t) = 0 \qquad \forall t \in [t_0, t_1]$$

$$\int_{t_0}^{t} \frac{\partial L}{\partial x}(s, x_*(s), \dot{x}_*(s), \lambda)\, ds - \frac{\partial L}{\partial \dot{x}}(t, x_*(t), \dot{x}_*(t), \lambda) + C = 0$$

Теор. док.

Пример 1 (Задача Дидоны)



Какой должна быть кривая, чтоб площадь была max

$l > 2T_0$

1-ая Задача Дидоны

$$l = \int_{-T}^{T} \sqrt{1 + \dot{x}^2(t)}\, dt$$

$$S = \int_{-T}^{T} x(t)\, dt \longrightarrow \max$$

$$\begin{cases} J(x) = -\int_{-T}^{T} x(t)\, dt \longrightarrow \min \\ \int_{-T}^{T} \sqrt{1 + \dot{x}^2(t)}\, dt = l \\ x(-T) = 0 = x(T) \qquad -T, T - \text{не фикс} \end{cases}$$

2-я Задача Дидоны

$L(t, x, \dot{x}, \lambda) = -\lambda_0 x + \lambda_1 \sqrt{1 + \dot{x}^2} \longleftarrow$ ф-ия Лагранжа

$$\left(\frac{\partial L}{\partial \dot{x}}(\cancel{t}, x_*(t), \dot{x}_*(t), \lambda), \dot{x}_*(t)\right) - L(x_*(t), \dot{x}_*(t), \lambda) = C$$

$$\frac{\partial L}{\partial \dot{x}} = \frac{\lambda_1 \dot{x}}{\sqrt{1 + \dot{x}^2}}$$

$$\frac{\lambda_1 \dot{x}^2(t)}{\sqrt{1 + \dot{x}^2(t)}} + \lambda_0 x(t) - \lambda_1 \sqrt{1 + \dot{x}^2(t)} = C_*$$

$$\frac{\lambda_1}{\sqrt{1 + \dot{x}^2(t)}} = \lambda_0 x(t) - C_*$$

$$\lambda_0 \neq 0 \qquad \lambda = \frac{\lambda_1}{\lambda_0} \qquad C_2 = \frac{C_*}{\lambda_0} \qquad \Rightarrow$$

$$\frac{\lambda}{\sqrt{1+\dot{x}^2(t)}} = x(t) - C_2$$

$$dx = \pm\sqrt{\frac{\lambda^2}{(x-C_2)^2} - 1}\, dt$$

$$\frac{(x-C_2)\,dx}{\sqrt{\lambda^2-(x-C_2)^2}} = \pm\, dt$$

$$\Rightarrow \boxed{(t-C_1)^2+(x-C_2)^2=\lambda^2}$$

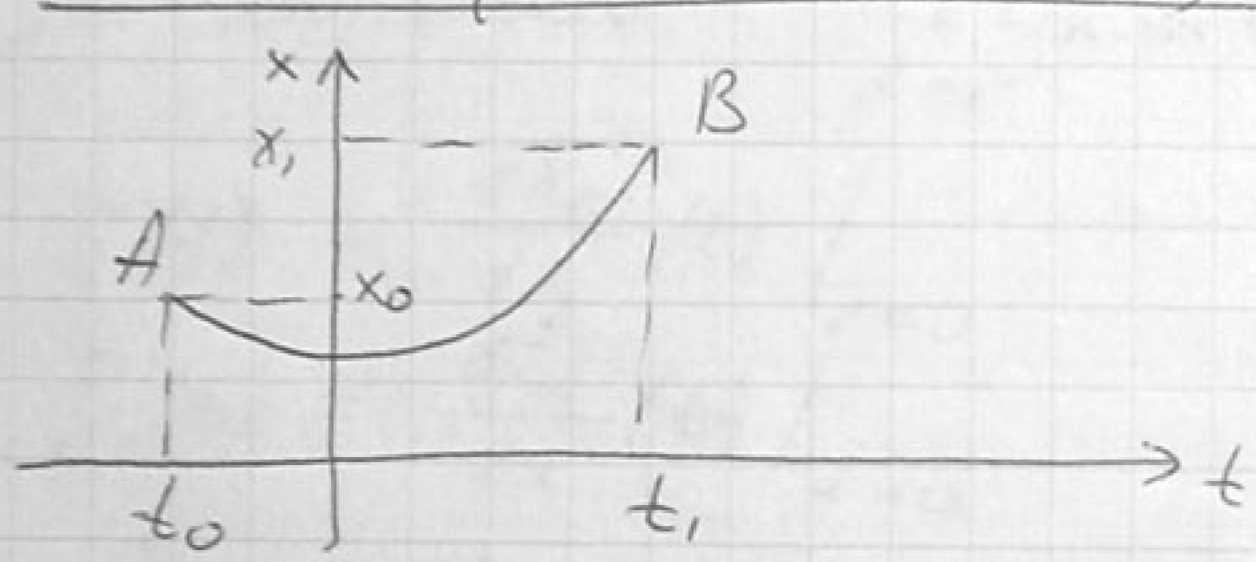
$$x(-T_0) = x(T_0) \Rightarrow C_1 = 0$$

ограничение

$$t^2+(x-C_2)^2 = |\lambda|^2 \qquad \boxed{T_0 = 1,\ l = \pi} \Rightarrow C_2 = 0,\ |x| = 1$$

$$\Rightarrow t^2 + x^2 = 1$$

Пример 2 (Цепная линия)



однородная, гибкая, цепь

H — длина цепи

Какую форму примет цепь, подвешенная в 2-х точках?

Потенциальная энергия $\to \min$

$$\begin{cases} J(x) = \int_{t_0}^{t_1} \underbrace{\rho g \sqrt{1+\dot{x}^2(t)} \cdot x(t)}_{f}\, dt \longrightarrow \min \\ x(t_0) = x_0, \quad x(t_1) = x_1 \\ \int_{t_0}^{t_1} \underbrace{\sqrt{1+\dot{x}^2}}_{h}\, dt = H \end{cases}$$

ф-ия Лагранжа $L(x, \dot{x}, \lambda) = \lambda_0 \rho g \sqrt{1+\dot{x}^2} \cdot x + \lambda_1 \sqrt{1+\dot{x}^2}$

$$H(x,\dot{x}) = \left(\frac{\partial L}{\partial \dot{x}}, \dot{x}\right) - L = Const$$

$$\rho g\, x(t) + \lambda = a\sqrt{1+\dot{x}^2(t)}$$

$$\frac{dx}{\sqrt{\left(\frac{\rho g x + \lambda}{a}\right)^2 - 1}} = \pm\, dt$$

$$\frac{d\left(\frac{\rho g x+\lambda}{a}\right)}{\sqrt{\underbrace{\left(\frac{\rho g x+\lambda}{a}\right)}_{ch\, u}^2-1}} \pm dt\cdot\frac{\rho g}{a}$$

$u = \pm\left(\frac{\rho g}{a}t+b\right)$ b – произв. постоянная

$$\frac{\rho g x+\lambda}{a} = ch\left(\pm\left[\frac{\rho g}{a}t+b\right]\right)$$

Вид экстремали $\boxed{x_*(t) = \frac{a}{\rho g}\, ch\left(\frac{\rho g}{a}t+b\right)-\frac{\lambda}{\rho g}}$

25.09.08 Задача Лагранжа

$$J(x) = \int_{t_0}^{t_1} f(t,x(t),\dot{x}(t))\,dt \to min$$

$x(t_0)=x_0,\ x(t_1)=x_1$

$g_i(t,x(t),\underbrace{\dot{x}(t)}_{\text{если нет}}) = 0 \quad i=\overline{1,k}$

$\Rightarrow$ голономные (нет $\dot{x}$)

$\Rightarrow$ неголономные (есть $\dot{x}$)

П1 Голономные связи

$$\begin{cases} J(x) = \int_{t_0}^{t_1} f(t,x(t),\dot{x}(t))\,dt \longrightarrow min \\ x(t_0)=x_0,\ x(t_1)=x_1 \quad (*) \\ g_i(t,x(t)) = 0 \quad i=\overline{1,k} \end{cases}$$

Пусть $k<n$

Теорема $\exists\ f(\cdot,\cdot,\cdot),\ g_i(\cdot,\cdot) \in C^1$

$\exists\ x_*(t)\in C^1[t_0,t_1]$ доставляет слабый лок min $(*)$

$\exists\ \left.\frac{D(g_1,\dots,g_k)}{D(x_1,\dots,x_k)}\right|_{x=x_*(t)} \neq 0 \quad t\in[t_0,t_1]$

$\boxed{g(t,x)=0}$ ~~ограниченная, [illegible]~~

Тогда $\exists$ $\overset{\text{такая ф-ия}}{\varphi(t)} = (\psi_1(t),\dots,\psi_k(t))$ такая, что

$L(t, \underset{*}{x}, \underset{*}{\dot{x}}, \psi) = f(t, \underset{*}{x}, \underset{*}{\dot{x}}) + \sum_{j=1}^{k} \psi_j(t)\, g_j(t, \underset{*}{x})$ удовлетворяет уравнению Эйлера

1 см

Док-во:

$$\exists\ x = (\bar{x}, \bar{\bar{x}}) - (\underbrace{x_1, \ldots, x_k}_{\bar{x}}, \underbrace{x_{k+1}, \ldots, x_n}_{\bar{\bar{x}}})$$

$\forall\ \bar{\bar{\eta}}(t) \in C^1[t_0, t_1]$

$\overset{\parallel}{(\bar{\bar{\eta}}_{k+1}(t), \bar{\bar{\eta}}_n(t))}$

$\bar{\eta}(t_0) = \bar{\eta}(t_1)$

$\bar{\bar{x}}_\alpha(t) = \bar{\bar{x}}_*(t) + \alpha \bar{\bar{\eta}}(t)$

$g(t, x) = 0$ — ограничение

$\bar{x} = G(t, \bar{\bar{x}}) \Rightarrow \bar{x}_\alpha(t) = G(t, \bar{\bar{x}}_\alpha(t))$

По построению $\underset{\alpha=0}{\bar{x}_*(t)} = G(t, \bar{\bar{x}}_*(t))$

$$\bar{\eta}(t) = \frac{d\bar{x}_\alpha}{d\alpha}(t)\Big|_{\alpha=0}$$

$$\dot{\bar{\eta}}(t) = \frac{d\dot{\bar{x}}_\alpha}{d\alpha}(t)\Big|_{\alpha=0}$$

$$\int_{t_0}^{t_1} \Big\{ \Big(\frac{\partial f}{\partial \bar{x}}(t, x_*(t), \dot{x}_*(t)), \bar{\eta}(t)\Big) + \Big(\frac{\partial f}{\partial \bar{\bar{x}}}(t, x_*(t), \dot{x}_*(t)), \bar{\bar{\eta}}(t)\Big) +$$
$$+ \Big(\frac{\partial f}{\partial \dot{\bar{x}}}(t, x_*(t), \dot{x}_*(t)), \dot{\bar{\eta}}(t)\Big) + \Big(\frac{\partial f}{\partial \dot{\bar{\bar{x}}}}(t, x_*(t), \dot{x}_*(t)), \dot{\bar{\bar{\eta}}}(t)\Big)\Big\} dt = 0$$

$g(t, x_\alpha(t)) = 0$

$$\frac{\partial g}{\partial \bar{x}}(t, x_*(t))\, \bar{\eta}(t) + \frac{\partial g}{\partial \bar{\bar{x}}}(t, x_*(t))\, \bar{\bar{\eta}}(t) = 0$$

возьмем $\forall$ ф-цию $\psi(t) \in C^1[t_0, t_1]$

$\Rightarrow$ сложим с $\int_{t_0}^{t_1}$

$$\Rightarrow \int_{t_0}^{t_1} \Big\{ \Big(\overbrace{\frac{\partial f}{\partial \bar{x}}(t, x_*(t), \dot{x}_*(t)) + \Big[\frac{\partial g}{\partial \bar{x}}(t, x_*(t))\Big]^T \psi(t)}^{\frac{\partial L}{\partial \bar{x}}}, \bar{\eta}(t)\Big) +$$
$$+ \Big(\overbrace{\frac{\partial f}{\partial \bar{\bar{x}}}(t, x_*(t), \dot{x}_*(t)) + \Big[\frac{\partial g}{\partial \bar{\bar{x}}}(t, x_*(t))\Big]^T \psi(t)}^{\frac{\partial L}{\partial \bar{\bar{x}}}}, \bar{\bar{\eta}}(t)\Big) +$$
$$+ \Big(\underbrace{\frac{\partial f}{\partial \dot{\bar{x}}}(t, x_*(t), \dot{x}_*(t))}_{\frac{\partial L}{\partial \dot{\bar{x}}}}, \dot{\bar{\eta}}(t)\Big) + \Big(\underbrace{\frac{\partial f}{\partial \dot{\bar{\bar{x}}}}(t, x_*(t), \dot{x}_*(t))}_{\cancel{\frac{\partial f}{\partial \dot{\bar{\bar{x}}}}}\ \frac{\partial L}{\partial \dot{\bar{\bar{x}}}}}, \dot{\bar{\bar{\eta}}}(t)\Big)\Big\} dt = 0$$

2 черты

Ф-ция Лагранжа $L(t, x, \dot{x}, t) = f(t, x, \dot{x}) + (g(t, x), \psi)$

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>>

$$\Rightarrow \int_{t_0}^{t_1} \Big\{ \Big(\frac{\partial L}{\partial \bar{x}}(t, x_*(t), \dot{x}_*(t)), \bar{\eta}(t)\Big) + \Big(\frac{\partial L}{\partial \bar{\bar{x}}}(t, x_*(t), \dot{x}_*(t), \psi(t)), \bar{\bar{\eta}}(t)\Big) +$$

$$+ \Big(\frac{\partial L}{\partial \dot{\bar{x}}}(t, x_*(t), \dot{x}_*(t), \psi(t)), \dot{\bar{\eta}}(t)\Big) +$$

$$+ \Big(\frac{\partial L}{\partial \dot{\bar{\bar{x}}}}(t, x_*(t), \dot{x}_*(t), \psi(t)), \dot{\bar{\bar{\eta}}}(t)\Big) \Big\} dt = 0$$

$$\Rightarrow \int_{t_0}^{t_1} \Big(\frac{\partial L}{\partial \bar{x}}(t, x_*(t), \dot{x}_*(t), \psi(t)), \bar{\eta}(t)\Big) dt = -$$

$$- \Big(\int_t^{t_1} \frac{\partial L}{\partial \bar{x}}(s, x_*(s), \dot{x}_*(s), \psi(s))\, ds - \bar{c}, \bar{\eta}(t)\Big)\Big|_{t_0}^{t_1} +$$

$$0$$

$$+ \int_{t_0}^{t_1} \Big(\int_t^{t_1} \frac{\partial L}{\partial \bar{x}}(s, x_*(s), \dot{x}_*(s), \psi(s))\, ds + \bar{c}, \dot{\bar{\eta}}(t)\Big) dt$$

$$g(t, x_*(t)) = 0$$

$$\frac{\partial g}{\partial \bar{x}} \bar{\eta} + \frac{\partial g}{\partial \bar{\bar{x}}} \bar{\bar{\eta}} = 0$$

$$\int_{t_0}^{t_1} \Big(\frac{\partial L}{\partial \bar{\bar{x}}}(t, x_*(t), \dot{x}_*(t), \psi(t)), \bar{\bar{\eta}}(t)\Big) dt =$$

$$= \int_{t_0}^{t_1} \Big(\int_t^{t_1} \frac{\partial L}{\partial \bar{\bar{x}}}(s, x_*(s), \dot{x}_*(s), \psi(s))\, ds + \bar{\bar{c}}, \dot{\bar{\bar{\eta}}}(t)\Big) dt \quad (**)$$

$$\Rightarrow \int_{t_0}^{t_1} \Big(\frac{\partial L}{\partial \dot{\bar{x}}}(t, x_*(t), \dot{x}_*(t), \psi(t)) + \int_{t_0}^{t_1} \frac{\partial L}{\partial \bar{x}}(s, x_*(s), \dot{x}_*(s), \psi(s))\, ds + \bar{c}, \dot{\bar{\eta}}(t)\Big) dt +$$

$$+ \int_{t_0}^{t_1} \Big(\underbrace{\frac{\partial L}{\partial \dot{\bar{\bar{x}}}}(t, x_*(t), \dot{x}_*(t), \psi(t)) + \int_t^{t_1} \frac{\partial L}{\partial \bar{\bar{x}}}(s, x_*(s), \dot{x}_*(s), \psi(s))\, ds + \bar{\bar{c}}}_{\dot{\bar{\bar{\eta}}}(t)}, \dot{\bar{\bar{\eta}}}(t)\Big) dt = 0$$

$$\boxed{L = L(t, x_*(t), \dot{x}_*(t), \psi(t))}$$

$$\psi(t): \quad \frac{\partial L}{\partial \dot{x}} + \int_t^{t_1} \Big(\frac{\partial L}{\partial x} \Big) ds = \text{const}$$

$$\psi(t) = \Big[\Big[\frac{\partial g}{\partial \bar{x}}(t, x_*(t)) \Big]^T \Big]^{-1} \Big\{ \frac{d}{dt} \frac{\partial f}{\partial \dot{\bar{x}}}(t, x_*(t), \dot{x}_*(t)) - \frac{\partial f}{\partial \bar{x}}(t, x_*(t), \dot{x}_*(t)) \Big\}$$

$$\overline{c} = -\frac{\partial L}{\partial \dot{\overline{x}}}(t, x_*(t_1), \dot{x}_*(t_1), \psi(t_1))$$

$$\Rightarrow (**) = 0$$

$$\Rightarrow \boxed{\psi(t): \ \frac{\partial L}{\partial \dot{\overline{x}}} + \int_t^{t_1} \frac{\partial L}{\partial \overline{x}} ds + \overline{c} = 0}$$

$$\cancel{\overline{\eta}(t) = \int_{t_0}^{t_1} \frac{\partial L}{\partial \dot{x}}(t, x_*(t)}$$

$$\overline{\overline{\eta}}(t) = \int_{t_0}^{t_1} \left\{ \frac{\partial L}{\partial \dot{\overline{\overline{x}}}} + \int_s^{t_1} \frac{\partial L}{\partial \overline{\overline{x}}} d\tau + \overline{\overline{c}} \right\} ds$$

$$\overline{\overline{\eta}}(t_0) = 0 = \overline{\overline{\eta}}(t_1)$$

$$\int_{t_0}^{t_1} (\overline{\overline{\eta}}(t), \dot{\overline{\overline{\eta}}}(t)) \, dt = 0$$

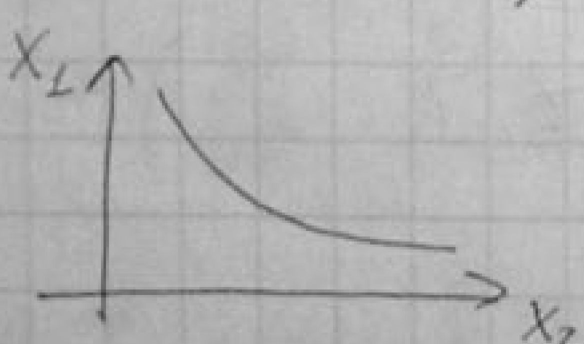
$$\Rightarrow \boxed{\frac{\partial L}{\partial \dot{\overline{\overline{x}}}} + \int_t^{t_1} \frac{\partial L}{\partial \overline{\overline{x}}} ds + \overline{\overline{c}} = 0}$$

$$c = (\overline{c}, \overline{\overline{c}})$$

Т. док

Пример Движение без трения мат точки по плоскому желобу

$$x_1 = \tau(x_2)$$



$$\begin{cases} J(x) = \int_{t_0}^{t_1} \left\{ m \dfrac{\dot{x}_1^2(t) + \dot{x}_2^2(t)}{2} - mg x_1(t) \right\} dt \to \min \\ g(x_1, x_2) = x_1 - \tau(x_2) = 0 \end{cases}$$

Должны проверить

$$\frac{\partial g}{\partial x_1} = 1 \neq 0$$

$$L(t,x,\dot{x},\psi) = m\frac{\dot{x}_1^2+\dot{x}_2^2}{2} - mg x_1 + \psi(x_1 - \tau(x_2))$$

$$\begin{cases} m\ddot{x}_1(t) - mg + \psi(t) = 0 \\ m\ddot{x}_2(t) - \psi(t)\tau'(x_2(t)) = 0 \end{cases}$$ Система ур-ний Эйлера

$$x_1(t) = \tau(x_2(t))$$

$$\Rightarrow \quad \dot{x}_1(t) = \tau'(x_2(t))\dot{x}_2(t)$$

$$\ddot{x}_1(t) = \tau''(x_2(t))\dot{x}_2^2(t) + \tau'(x_2(t))\ddot{x}_2(t)$$

$$\begin{cases} 0 = \ddot{x}_2(t)\left(1+\left[\tau'(x_2(t))\right]^2\right) + \tau'(x_2(t))\tau''(x_2(t))(\dot{x}_2(t))^2 + g\tau'_{(x_2(t))} \\ x_1(t) = \tau(x_2(t)) \end{cases}$$

П2 Неголономные связи

$$\begin{cases} J(x) = \int_{t_0}^{t_1} f(t, x(t), \dot{x}(t))\,dt \to \min \\ x(t_0) = x_0 \\ g_i(t, x(t), \dot{x}(t)) = 0 \quad i=\overline{1,k},\ k<n \end{cases}$$

$$f, g \in C^1(\cdot,\cdot,\cdot)$$

Теорема

$\exists\ x_*(t) \in C^1[t_0,t_1]$ доставл. слаб. лок. min в $(*)$

$$\exists \left.\frac{D(g_1,\dots,g_k)}{D(\dot{x}_1,\dots,\dot{x}_k)}\right|_{\substack{x=x_*(t)\\ \dot{x}=\dot{x}_*(t)}} \neq 0, \ \forall t \in [t_0,t_1]$$

Тогда найдется $\psi(t) = (\psi_1(t),\dots,\psi_k(t)) \neq 0$

$$\boxed{L(t, x_*(t), \dot{x}_*(t), \psi(t)) = L, \quad g(t, x_*(t), \dot{x}_*(t)) = g, \quad f(\ \text{—''—}\ ) = f}$$

$$L = f + \sum_{j=1}^{k}\psi_j(t)\,g_j, \text{ то}$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \leftarrow \text{Ур-ние Эйлера}$$

$$\left[\frac{\partial L}{\partial \dot{\bar x}}(t_1, x_*(t_1), \dot x_*(t_1), \psi(t_1)) = 0\right.$$

Разобьем вектор x на 2 куска $\left(x = (\bar x, \bar{\bar x})\right)$

$$\bar{\bar\eta}(t) \in C^1[t_0, t_1]$$

$$\| \quad (\bar{\bar\eta}_{k+1}, \dots, \bar{\bar\eta}_n(t)), \quad \bar{\bar\eta}(t_0) = 0$$

$$\bar{\bar x}_\alpha(t) = \bar{\bar x}_*(t) + \alpha\bar{\bar\eta}(t)$$

$$g(t, x, \dot x) = 0$$

$$\dot{\bar x} = G(t, x, \dot{\bar{\bar x}}) \leftarrow$$ по Т. о неявной ф-ии

$$\begin{cases} \dot{\bar x}_\alpha(t) = G(t, x_\alpha(t), \dot{\bar{\bar x}}_\alpha(t)) \\ \bar x_\alpha(t_0) = \bar x_0 \end{cases}$$
$\leftarrow$ обеспечивает $\exists$-ние решение на заданном отрезке.

$$\bar x_\alpha(t)\Big|_{\alpha=0} = \bar x_*(t)$$

$$\bar\eta(t) = \frac{d\bar x_\alpha}{d\alpha}(t)\Big|_{\alpha=0}$$

$$\dot{\bar\eta}(t) = \frac{d\dot{\bar x}_\alpha}{d\alpha}(t)\Big|_{\alpha=0}$$

$$J = \int_{t_0}^{t_1} \left\{ \left(\frac{\partial f}{\partial \bar x}(t, x_*(t), \dot x_*(t)), \bar\eta(t)\right) + \left(\frac{\partial f}{\partial \bar{\bar x}}, \bar{\bar\eta}(t)\right) + \left(\frac{\partial f}{\partial \dot{\bar x}}, \dot{\bar\eta}(t)\right) + \right.$$
$$\left. + \left(\frac{\partial f}{\partial \dot{\bar{\bar x}}}, \dot{\bar{\bar\eta}}(t)\right) \right\} dt$$

$$g(t, x_\alpha(t), \dot x_\alpha(t)) = 0$$

Дифф и кладем $\alpha = 0$

$$\left[\frac{\partial g}{\partial \bar x}(t, x_*(t), \dot x_*(t))\right]^* \bar\eta(t) + \left[\frac{\partial g}{\partial \bar{\bar x}}\right]^* \bar{\bar\eta}(t) + \left[\frac{\partial g}{\partial \dot{\bar x}}\right]^* \dot{\bar\eta}(t) +$$
$$+ \left[\frac{\partial g}{\partial \dot{\bar{\bar x}}}\right]^* \dot{\bar{\bar\eta}}(t) = 0$$

Возьмем $\forall$ ф-ию $\psi(t) \in C^1[t_0, t_1]$

Делаем как в предыдущ. теореме

$$\frac{\partial L}{\partial \bar{x}} = \frac{\partial f}{\partial \bar{x}} + \left[\frac{\partial g}{\partial \bar{x}}\right]^T \psi$$

$$L(t, x, \dot{x}, \psi) = f(t, x, \dot{x}) + \sum_{j=1}^{k} \psi_j g_j(t, x, \dot{x})$$

$$0 = \int_{t_0}^{t} \left\{ \left(\frac{\partial L}{\partial \bar{x}}, \bar{\eta}(t)\right) + \left(\frac{\partial L}{\partial \bar{\bar{x}}}, \bar{\bar{\eta}}(t)\right) + \left(\frac{\partial L}{\partial \dot{\bar{x}}}, \dot{\bar{\eta}}(t)\right) + \left(\frac{\partial L}{\partial \dot{\bar{\bar{x}}}}, \dot{\bar{\bar{\eta}}}(t)\right) dt\right.$$

$$\int_{t_0}^{t_1} \left(\frac{\partial L}{\partial \bar{x}}, \bar{\eta}(t)\right) dt = \cancel{-\left(\int_t^{t_1} \frac{\partial L}{\partial \bar{x}} ds, \eta(t)\right)\Big|_{t_0}^{t_1}} \to 0 \; +$$

$$+ \int_{t_0}^{t_1} \left(\int_t^{t_1} \frac{\partial L}{\partial \bar{x}} ds, \dot{\bar{\eta}}(t)\right) dt$$

$$\Rightarrow \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial \bar{x}}, \bar{\eta}(t)\right) dt = \int_{t_0}^{t_1} \left(\int_t^{t_1} \frac{\partial L}{\partial \bar{x}} ds, \dot{\bar{\eta}}(t)\right) dt$$

$$\Rightarrow \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial \bar{\bar{x}}}, \bar{\bar{\eta}}(t)\right) dt = \int_{t_0}^{t_1} \left(\int_t^{t_1} \frac{\partial L}{\partial \bar{\bar{x}}} ds, \dot{\bar{\bar{\eta}}}(t)\right) dt$$

$$\Rightarrow \int_{t_0}^{t_1} \Big(\underbrace{\frac{\partial L}{\partial \dot{\bar{x}}} + \int_t^{t_1} \frac{\partial L}{\partial \bar{x}} ds}_{\text{надо, чтобы } =0}, \dot{\bar{\eta}}(t)\Big) dt + \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial \dot{\bar{\bar{x}}}} + \int_t^{t_1} \frac{\partial L}{\partial \bar{\bar{x}}} ds, \dot{\bar{\bar{\eta}}}(t)\right) dt = 0$$

$$\psi(t): \dot{\psi}(t) = \left[\left[\frac{\partial g}{\partial \dot{\bar{x}}}\right]^T\right]^{-1} \left\{\frac{\partial g}{\partial \bar{x}} - \frac{d}{dt}\frac{\partial g}{\partial \dot{\bar{x}}}\right\} \psi(t) + \left[\left[\frac{\partial g}{\partial \dot{\bar{x}}}\right]^T\right]^{-1} \cdot$$

$$\cdot \left\{\frac{\partial f}{\partial \bar{x}} - \frac{d}{dt}\frac{\partial f}{\partial \dot{\bar{x}}}\right\}$$

При $t = t_1$ $\boxed{\frac{\partial L}{\partial \dot{\bar{x}}}\Big|_{t=t_1} = 0}$ (**) $+$ $\boxed{\frac{\partial L}{\partial \dot{\bar{x}}} + \int_t^{t_1} \frac{\partial L}{\partial \bar{x}} ds = 0}$

Т.о. ф-ию $\psi(t)$ мы подобрали

$$\Rightarrow \bar{\bar{\eta}}(t) = \int_{t_0}^{t} \left\{\frac{\partial L}{\partial \dot{\bar{\bar{x}}}} + \int_t^{t_1} \frac{\partial L}{\partial \bar{\bar{x}}} d\tau\right\} ds$$

$$\bar{\bar{\eta}}(t_0) = 0$$

$$\int_{t_0}^{t_1} (\dot{\bar{\bar{\eta}}}(t), \dot{\bar{\bar{\eta}}}(t)) dt = 0 \quad \Rightarrow$$

$$\dot{\bar{\bar{\eta}}}(t) \equiv 0$$

$$\frac{\partial L}{\partial \dot{\bar{\bar{x}}}} + \int_t^{t_1} \frac{\partial L}{\partial \bar{\bar{x}}} ds = 0$$

$$\Rightarrow \boxed{\frac{\partial L}{\partial \dot{\bar{\bar{x}}}} = 0} \quad ****$$

Объединим $*** + *****$ $\boxed{\frac{\partial L}{\partial \dot{x}} = 0}$

$$\Rightarrow \boxed{\frac{\partial L}{\partial \dot{x}} + \int_t^{t_1} \frac{\partial L}{\partial x} ds = 0}$$ ур-ние Эйлера

Пример Линейно-квадратичная задача

$$J(u) = \frac{1}{2}\int_{t_0}^{t_1} \{ (Qy(t), y(t)) + (Ru(t), u(t)) \} dt \to \min$$

$$\dot{y}(t) = Ay(t) + Bu(t)$$

$$y(t_0) = y_0$$

$$y \in \mathbb{R}^k, \ u \in \mathbb{R}^l$$

$Q = Q^T > 0$, $R = R^T > 0$ матрицы

Замена $x = (\bar{x}, \bar{\bar{x}})$ размерности $k + l = n > k$

$$\bar{x}(t) = y(t)$$

$$\bar{\bar{x}}(t) = \int_{t_0}^{t_1} u(s) ds$$

$$\Rightarrow \begin{cases} J(x) = \frac{1}{2}\int_{t_0}^{t_1} \{ (Q\bar{x}(t), \bar{x}(t)) + (R\dot{\bar{\bar{x}}}(t), \dot{\bar{\bar{x}}}(t)) \} dt \to \min \\ x(t_0) = x_0 = (y_0, 0) \\ \dot{\bar{x}}(t) - A\bar{x}(t) - B\dot{\bar{\bar{x}}}(t) = 0 \end{cases}$$

$$L(t, x, \dot{x}, \psi) = \frac{1}{2}(Q\bar{x}, \bar{x}) + \frac{1}{2}(R\dot{\bar{\bar{x}}}, \dot{\bar{\bar{x}}}) + (\psi, \dot{\bar{x}} - A\bar{x} - B\dot{\bar{\bar{x}}})$$

$$\frac{\partial L}{\partial \bar{x}} = Q\bar{x} - A^T\psi$$

$$\frac{\partial L}{\partial \bar{\bar{x}}} = 0$$

$$\frac{\partial L}{\partial \dot{x}} = \psi$$

$$\frac{\partial L}{\partial \dot{\bar{x}}} = R\dot{\bar{x}} - B^T\psi$$

$$\begin{cases} \dot{\psi}(t) = -A^T\psi + Q\bar{x}(t) \\ \dfrac{d}{dt}\left(R\dot{\bar{x}}(t) - B^T\psi(t)\right) = 0 \end{cases}$$

$$\begin{cases} \psi(t_1) = 0 \\ R\dot{\bar{x}}(t_1) - B^T\psi(t_1) = 0 \end{cases}$$

$$\to R\dot{\bar{x}}(t) = B^T\psi(t)$$

$$\begin{cases} \dot{\psi}(t) = -A^T\psi(t) + Q\bar{x}(t) \\ \psi(t_1) = 0 \end{cases}$$

$$x_*(t) = (\bar{x}_*(t), \dot{\bar{x}}_*(t))$$

$$L(t, x_*(t), \dot{x}_*(t), \psi(t)) = \min_{\dot{\bar{x}}} L(t, x_*(t), \dot{\bar{x}}_*(t), \dot{\bar{x}}, \psi(t))$$

$\Rightarrow u_*(t) = R^{-1}B^T\psi(t) \leftarrow$ формула для опт. упр.

$$\begin{cases} \dot{\psi}(t) = -A^T\psi(t) + Qy_*(t) \\ \psi(t_1) = 0 \end{cases}$$

$$\begin{cases} \dot{y}_*(t) = Ay_*(t) + Bu_*(t) \\ y_*(t_0) = y_0 \end{cases}$$

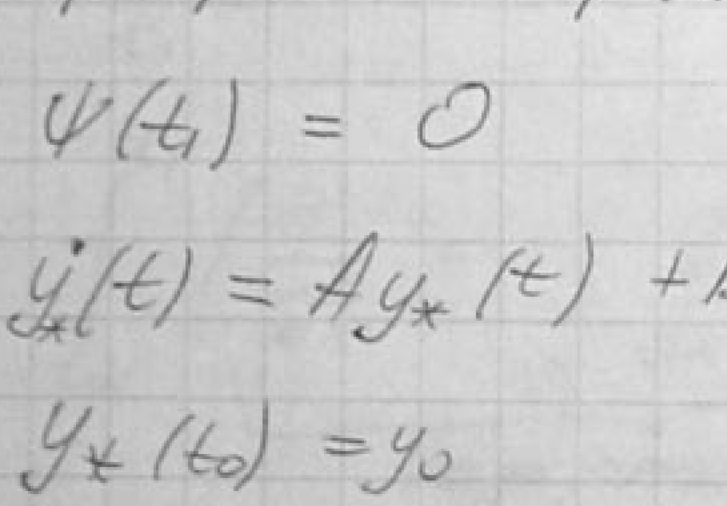
Краевая задача принципа max для ОУ

§ Случай негладких экстремалей

Условие Вейерштрасса – Эрдмана

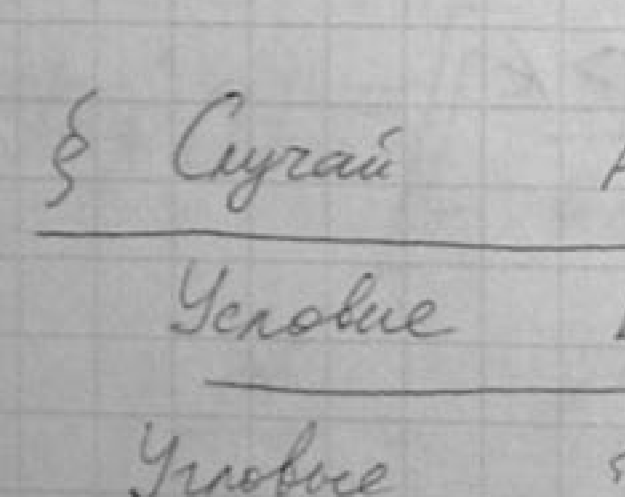
Угловые точки — производная терпит разрыв I рода

Задача о нахождении кривой, образующей при своем вращении поверхность возможно меньшей площади



$$J(x) = \int_{-a}^{a} x\sqrt{1+\dot{x}^2}\,dt \to \min$$

$x(-a) = A$
$x(a) = A \qquad a>0, A>0$



$H = x\sqrt{1+\dot{x}^2} - x\cdot\dfrac{\dot{x}^2}{\sqrt{1+\dot{x}^2}} = C$ $\qquad$ $H = \dfrac{\partial f}{\partial \dot{x}}\dot{x} - f = C$

$$x = C\sqrt{1+\dot{x}^2}$$

$$\dot{x} = \pm\sqrt{\frac{x^2-c^2}{c^2}}$$

$$\int\frac{C\,dx}{\sqrt{x^2-c^2}} = \int dt$$

$$\Downarrow$$

$$t + c_1 = C\ln\left(x+\sqrt{x^2-c^2}\right)$$

$$x(t) = C\cdot \operatorname{ch}\frac{t+c}{c}$$

$$\Rightarrow \operatorname{ch}\frac{a+c_1}{c} = \operatorname{ch}\frac{-a+c_1}{c}$$

$$\frac{a+c_1}{c} = \frac{-a+c_1}{c} \qquad\qquad \frac{a+c_1}{c} = \frac{a-c_1}{c}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$a=0 \qquad\qquad\qquad\qquad c_1 = 0$$

$$\varnothing$$

Экстремали задачи: $x(t) = C\operatorname{ch} t/C$

$$\frac{A}{C} = \operatorname{ch}\frac{a}{C}$$

Замена:

$$z = \frac{a}{c},\ \alpha = \frac{A}{a} \Rightarrow \alpha z = \operatorname{ch} z,\ z>0$$

Расм. $\varphi(z) = \operatorname{ch} z - \alpha z$

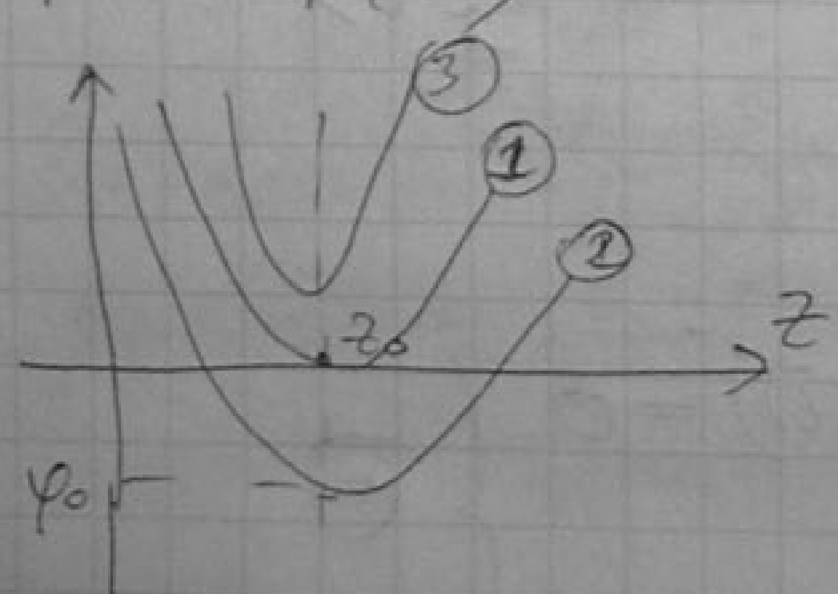
$\varphi'(z) = \operatorname{sh} z - \alpha = 0 \Rightarrow z_0 = \ln\left(\alpha+\sqrt{\alpha^2+1}\right)$ min

$\varphi''(z) = \operatorname{ch} z > 0$

$\varphi'(+0) < 0$

$\varphi'(z) \to +\infty \big/_{z\to+\infty}$

$\varphi_0 = \varphi(z_0)$



$\varphi_0 = 0$

1. корень 1 $\Rightarrow$ единств. $C_0 \Rightarrow$ $\exists$ единств. кривые $x_0^*(t) = C_0\operatorname{ch}\frac{t}{C_0}$

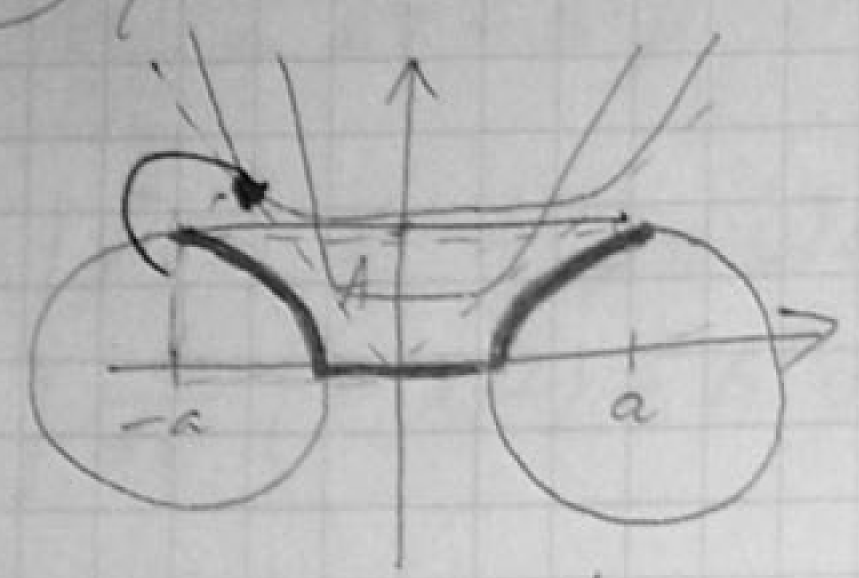
2. $\varphi_0 < 0 \to$ корней 2

$z_1 < z_0 < z_2$

$\Downarrow$

$\exists c_1, c_2 \Rightarrow \exists x_1^*(t) = c_1\operatorname{ch}\frac{t}{c_1}$, $x_2^*(t) = c_2\operatorname{ch} t/c_2$

③ $\varphi_0 > 0 \to$ корней нет



$$\begin{cases} J(x) = \int_{t_0}^{t_1} f(t, x(t), \dot x(t))\,dt \to \min \\ x(t_0) = x_0, \quad x(t_1) = x_1 \end{cases} \qquad (*)$$

<u>Лемма</u> (Теорема Гильберта о дифф)

] $x_*(t)$, $t \in [t_0, t_1]$ – экстремаль в (*)

] $f(t, x, \dot x) \in C^2$ Тогда во всех точках

$t \in (t_0, t_1)$ $\frac{\partial^2 f}{\partial \dot x^2}(t, x_*(t), \dot x(t)) \neq 0$ д Ф-ие $x_*(t) \in C^2$

дважды непр. дифф.

7.10.08

□ <u>Док-во</u>: ($\Rightarrow$) $\forall \tilde t \in (t_0, t_1)$

$$\frac{\partial^2 f}{\partial \dot x^2}(t, x_*(t), \dot x_*(t)) \neq 0$$

$\Downarrow$

$\exists \Delta : \tilde t \in \Delta$ в которой (*) выполнено

$x_*(t)$ – решение ур-ние Эйлера

$\Rightarrow \forall t \in \Delta$ $\quad f = f(t, x_*(t), \dot x_*(t))$

$$\frac{\partial f}{\partial \dot x}(t, x_*(t), \dot x_*(t)) = \frac{\partial f}{\partial \dot x}(\tilde t, x_*(\tilde t), \dot x_*(\tilde t)) + \int_{\tilde t}^{t} \frac{\partial f}{\partial x}\, ds$$

Рассм. малую окр-ть Λ точки $\dot x_*(\tilde t)$

] $F: R^1 \times R^n \to R^n$ на $\Delta \times \Lambda$

$$F(t, z) = \frac{\partial f}{\partial \dot x}(t, x_*(t), z) - \frac{\partial f}{\partial \dot x}(\tilde t, x_*(\tilde t), \dot x_*(\tilde t)) - \int_{\tilde t}^{t} \frac{\partial f}{\partial x}\, ds$$

$\Downarrow$ по условию Леммы – непр - дифф

$C^1(\Delta \times \Lambda)$

$F(t, z) = 0 : \ \forall \tilde t \in \Delta \quad F(\tilde t, \dot x_*(\tilde t)) = 0$ $\Big\}$ (*)

$\frac{\partial F}{\partial z}(\bar t, \dot x_*(\bar t)) \neq 0$]

просто / не замыкание

(*) По теореме о неявной ф-ции $\exists\ \bar\Delta \subset \Delta$ точки $\bar t$, в которой при условии $z(\bar t) = \dot x_*(\bar t)$ однозначно разрешимо и его решение $z(t) \in C^1(\bar\Delta)$

~~$\dot x_*(t)$ решение ... $\bar\Delta$ – решение~~

$\dot x_*(t)$ – решение в окр-ти $\bar\Delta$ у ур-ния $F(t,z)=0$

$\Rightarrow z(t) = \dot x_*(t),\ t \in \bar\Delta \Rightarrow \dot x_*(t) \in C^1(\bar\Delta)$

Вывод: экстремаль $x_*(t)$ может иметь угловые точки только там, где $\frac{\partial^2 f}{\partial \dot x^2}(t, x_*(t), \dot x_*(t))$ вырождена

<u>Теорема</u> (Условие Вейерштрасса – Эрдмана)

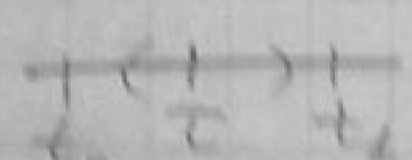
Пусть $x_*(t),\ t \in [t_0, t_1]$ – кусочно-дифф. ф-ция, доставляющая лок. min и $\tau \in (t_0, t_1)$ – угловая точка. Тогда

~~$\frac{\partial f}{\partial \dot x}(\tau, x_*(\tau), \dot x_*(\tau))$~~

1) $\frac{\partial f}{\partial \dot x}(\tau, x_*(\tau), \dot x_*(\tau-0)) = \frac{\partial f}{\partial \dot x}(\tau, x_*(\tau), \dot x_*(\tau+0))$

2) $f(\tau, x_*(\tau), \dot x_*(\tau-0)) - \left(\frac{\partial f}{\partial \dot x}(\tau, x_*(\tau), \dot x_*(\tau-0)), \dot x_*(\tau-0)\right) =$
$= f(\tau, x_*(\tau), \dot x_*(\tau+0)) - \left(\frac{\partial f}{\partial \dot x}(\tau, x_*(\tau), \dot x_*(\tau+0)), \dot x_*(\tau+0)\right)$

□ <u>Док-во</u>: $\forall \Delta \in R^n$ и δ: $\tau - \delta > t_0,\ \tau + \delta < t_1,\ \delta^2 + \|\Delta\|^2 \neq 0$



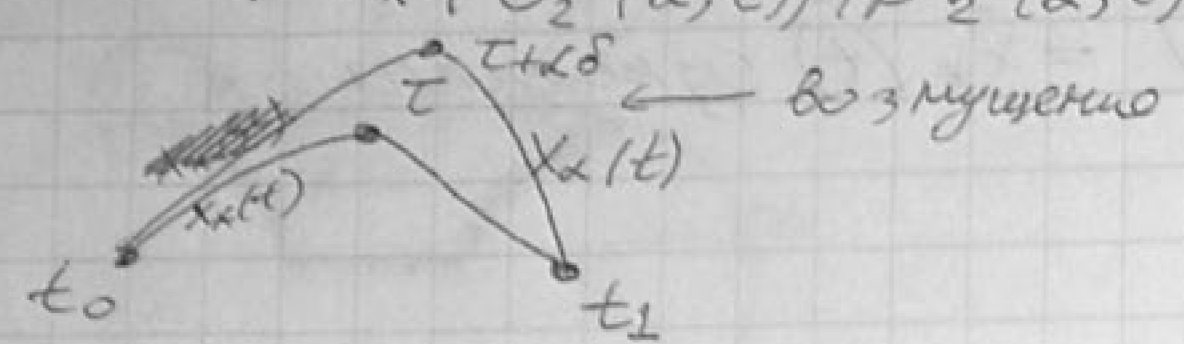
угловые точки $\left\{ \begin{array}{l} x_*(\tau-0) = x_*(\tau+0) \\ \dot x_*(\tau-0) \neq \dot x_*(\tau+0) \end{array} \right.$

$\left. \begin{array}{l} \gamma_1(\alpha,t) = \frac{\tau - t_0}{\tau + \alpha\delta - t_0} \cdot t + \frac{\alpha\delta}{\tau + \alpha\delta - t_0} \\ \beta_1(\alpha,t) = \frac{t - t_0}{\tau + \alpha\delta - t_0}\alpha \end{array} \right] t \in [t_0, \tau + \alpha\delta]$

$\left. \begin{array}{l} \gamma_2(\alpha,t) = \frac{t_1 - \tau}{t_1 - \alpha\delta - \tau} t - \frac{\alpha\delta}{t_1 - \alpha\delta - \tau} t_1 \\ \beta_2(\alpha,t) = \frac{t_1 - t}{t_1 - \alpha\delta - \tau}\alpha \end{array} \right] t \in [\tau + \alpha\delta, t_1]$

Построим семейство ф-ций

$$x_\alpha = \begin{cases} x_*(\gamma_1(\alpha,t)) + \beta_1(\alpha,t)\Delta & t_0 \le t \le \tau + \alpha\delta \\ x_*(\gamma_2(\alpha,t)) + \beta_2(\alpha,t)\Delta, & \tau+\alpha\delta \le t \le t_1 \end{cases}$$



$$x_0(t) = x_*(t)$$

$$\begin{cases} x_\alpha((\tau+\alpha\delta)-0) = x_\alpha((\tau+\alpha\delta)+0) \\ \dot{x}_\alpha((\tau+\alpha\delta)-0) \ne \dot{x}_\alpha((\tau+\alpha\delta)+0) \end{cases}$$ новая угловая точка (других угловых точек нет

Подставляем в ф-ал.

$$I(\alpha) = J(x_\alpha) = \int_{t_0}^{t_1} f(t, x_\alpha(t), \dot{x}_\alpha(t))\,dt \quad \text{лок min}$$

$$\dot{I}(0) = 0$$

Рассм. случай

$\alpha\delta > 0$

$$I(\alpha) - I(0) = \int_{t_0}^{\tau} \Big\{ f\Big(t, x_*(\underline{\underline{\gamma_1}}(\alpha,t)) + \underline{\underline{\beta_1}}(\alpha,t)\Delta, \dot{x}_*(\underline{\underline{\gamma_1}}(\alpha,t))\frac{\partial\gamma_1}{\partial t}(\alpha,t) + \frac{\partial\beta_1}{\partial t}(\alpha,t)\Delta\Big) - f(t, x_*(t), \dot{x}_*(t))\Big\}\,dt +$$

$$+ \int_{\tau}^{\tau+\alpha\delta} \Big\{ f\Big(t, x_*(\underline{\underline{\gamma_1}}(\alpha,t)) + \underline{\underline{\beta_1}}(\alpha,t)\Delta, \dot{x}_*(\underline{\underline{\gamma_1}}(\alpha,t))\frac{\partial\gamma_1}{\partial t}(\alpha,t) + \frac{\partial\beta_1}{\partial t}(\alpha,t)\Delta\Big) - f(t, x_*(t), \dot{x}_*(t))\Big\}\,dt +$$

$$+ \int_{\tau+\alpha\delta}^{t_1} \Big\{ f\Big(t, x_*(\underline{\underline{\gamma_2}}(\alpha,t)) + \underline{\underline{\beta_2}}(\alpha,t)\Delta, \dot{x}_*(\underline{\underline{\gamma_2}}(\alpha,t))\frac{\partial\gamma_2}{\partial t}(\alpha,t) + \frac{\partial\beta_2}{\partial t}(\alpha,t)\Delta\Big) - \text{—''—}\Big\}\,dt \ominus$$

$$\int_a^b f(x)\,dx = f(a)(b-a) + \underline{\quad}$$

$$\ominus \big\{ f(\tau, x_*(\tau), \dot{x}_*(\tau-0)) - f(\tau, x_*(\tau), \dot{x}_*(\tau+0)) \big\}\,\alpha\delta +$$

$$+ \int_{t_0}^{\tau} \Big\{ \Big(\frac{\partial f}{\partial x}(t, x_*(t), \dot{x}_*(t)), x_*(\gamma_1(\alpha,t)) + \beta_1(\alpha,t)\Delta - x_*(t)\Big) +$$

$$+ \Big(\frac{\partial f}{\partial \dot{x}}(\text{ —''— }), \dot{x}_*(\gamma_1(\alpha,t)) \cdot \frac{\partial\gamma_1}{\partial t}(\alpha,t)) + \text{—''—} - \dot{x}_*(t)\Big) \oplus$$

Ф-ла конечных приращений

$$f(x+\Delta x, y+\Delta y) - f(x,y) = \Big(\frac{\partial f}{\partial x}(x,y), \Delta x\Big) + \Big(\frac{\partial f}{\partial y}(x,y), \Delta y\Big) + \underline{\quad}$$

$\oplus$

$$\oplus \int_{\tau+\alpha\delta}^{t_1} \Big\{ \Big(\frac{\partial f}{\partial x}(t, x_*(t), \dot x_*(t)), x_*(z_2(\alpha,t)) + \beta_2(\alpha,t)\Delta - x_*(t) \Big) +$$

$$+ \Big(\frac{\partial f}{\partial \dot x}(t, x_*(t), \dot x_*(t)), \dot x_*(z_2(\alpha,t)) \frac{\partial z_2}{\partial t}(\alpha,t) +$$

$$+ \frac{\partial \beta_2}{\partial t}(\alpha,t)\Delta - \dot x_*(t) \Big) \Big\} dt + o(\alpha)$$

$$I(\alpha) - I(0) = \{ f(\tau, x_*(\tau), \dot x_*(\tau - 0)) - f(\tau, x_*(\tau), \dot x_*(\tau+0)) \} \alpha\delta +$$

$$+ \Big(\frac{\partial f}{\partial \dot x}(t, x_*(t), \dot x_*(t)), x_*(z_1(\alpha,t)) + \beta_1(\alpha,t)\Delta - x_*(t) \Big) \Big|_{t_0}^{\tau-0} +$$

$$+ \int_{t_0}^{\tau} \Big(\frac{\partial f}{\partial x}(t, x_*(t), \dot x_*(t)) - \frac{d}{dt}\frac{\partial f}{\partial \dot x}(t, x_*(t), \dot x_*(t)), x_*(z_1(\alpha,t)) + \beta_1(\alpha,t)\Delta - x_*(t) \Big) dt +$$

$$+ \Big(\frac{\partial f}{\partial \dot x}(t, x_*(t), \dot x_*(t)), x_*(z_2(\alpha,t)) + \beta_2(\alpha,t)\Delta - x_*(t) \Big) \Big|_{(\tau+\alpha\delta)+0}^{t_1} +$$

$$+ \int_{\tau+\alpha\delta}^{t_1} \Big(\frac{\partial f}{\partial x}(t, x_*(t), \dot x_*(t)) - \frac{d}{dt}\frac{\partial f}{\partial \dot x}(t, x_*(t), \dot x_*(t)), x_*(z_2(\alpha,t)) + \beta_2(\alpha,t)\Delta - x_*(t) \Big) dt + o(\alpha)$$

$[t_0, \tau]$ $[\tau, t_1]$

$x_*(t)$ удовлетворяет ур-нию Эйлера (t_0, x_0) и $(\tau, x_*(\tau))$
$(\tau, x_*(\tau))$ и (t_1, x_1)

$$x_*(z_1(\alpha, t_0)) + \beta_1(\alpha, t_0)\Delta - x_*(t_0) = 0$$

$$x_*(z_2(\alpha, t_1)) + \beta_2(\alpha, t_1)\Delta - x_*(t_1) = 0$$

$$\Rightarrow I(\alpha) - I(0) = \{ f(\tau, x_*(\tau), \dot x_*(\tau-0)) - f(\tau, x_*(\tau), \dot x_*(\tau+0)) \} \alpha\delta +$$

$$+ \Big(\frac{\partial f}{\partial \dot x}(\tau, x_*(\tau), \dot x_*(\tau-0)), x_*(z_1(\alpha, \tau-0)) + \beta_1(\alpha,\tau)\Delta - x_*(\tau-0) \Big) -$$

$$- \Big(\frac{\partial f}{\partial \dot x}(\tau+\alpha\delta, x_*(\tau+\alpha\delta), \dot x_*((\tau+\alpha\delta)+0)), x_*(\tau+\delta) + \alpha\Delta -$$

$$- x_*((\tau+\alpha\delta)+0) \Big) + o(\alpha) =$$

в остальном меняем местами

$$\Rightarrow I(\alpha) - I(0) = \{ \text{тоже самое} \} \alpha\delta +$$

$$+ \Big(\frac{\partial f}{\partial \dot x}(\tau, x_*(\tau), \dot x_*(\tau-0)), x_*(z_1(\alpha, \tau-0)) - x_*(\tau-0) + \beta_1(\alpha,\tau)\Delta \Big) -$$

$$- \Big(\frac{\partial f}{\partial \dot x}(\tau, \overset{x_*(\tau)}{\vee} \dot x_*(\tau+0)), -x_*((\tau+\alpha\delta)+0) + x_*(\tau+0) + \alpha\Delta \Big) + o_1(\alpha)$$

$$\lim_{\substack{\alpha\to 0\\ \alpha\delta>0}} \frac{1}{\alpha}\left(x_*(\beta_1(\alpha,\tau-0)) - x_*(\tau-0)\right) = -\delta \dot{x}_*(\tau-0)$$

$$\lim_{\substack{\alpha\to 0\\ \alpha\delta>0}} \frac{1}{\alpha}\left(x_*((\tau+\alpha\delta)+0) - x_*(\tau+0)\right) = \delta \dot{x}_*(\tau+0)$$

$$\lim_{\substack{\alpha\to 0\\ \alpha\delta>0}} \frac{\beta_1(\alpha,\tau)}{\alpha} = 1$$

$$\dot{I}(0)\Big|_{\alpha\delta>0} = \lim_{\substack{\alpha\to 0\\ \alpha\delta>0}} \frac{I(\alpha)-I(0)}{\alpha} = \Big(\frac{\partial f}{\partial \dot{x}}(\tau, x_*(\tau), \dot{x}_*(\tau-0)) - \frac{\partial f}{\partial \dot{x}}(\tau, x_*(\tau), \dot{x}_*(\tau+0)), \Delta\Big) +$$

$$+ \Big[\Big\{f(\tau, x_*(\tau), \dot{x}_*(\tau-0)) - \Big(\frac{\partial f}{\partial \dot{x}}(\tau, x_*(\tau), \dot{x}_*(\tau-0)), \dot{x}_*(\tau-0)\Big)\Big\} -$$

$$- \Big\{f(\tau, x_*(\tau), \dot{x}_*(\tau+0)) - \Big(\frac{\partial f}{\partial \dot{x}}(\tau, x_*(\tau), \dot{x}_*(\tau+0)), \dot{x}_*(\tau+0)\Big)\Big\}\Big]\delta$$

9.10.08

Аналогично для $\alpha\delta<0$

$$0 = \dot{I}(0) = \Big(\frac{\partial f}{\partial \dot{x}}(\tau, x_*(\tau), \dot{x}_*(\tau-0)) - \frac{\partial f}{\partial \dot{x}}(\tau, x_*(\tau), \dot{x}_*(\tau+0)), \Delta\Big) +$$

$$+ \Big[\Big\{f(\tau, x_*(\tau), \dot{x}_*(\tau-0)) - \Big(\frac{\partial f}{\partial \dot{x}}(\tau, x_*(\tau), \dot{x}_*(\tau-0)), \dot{x}_*(\tau-0)\Big)\Big\} -$$

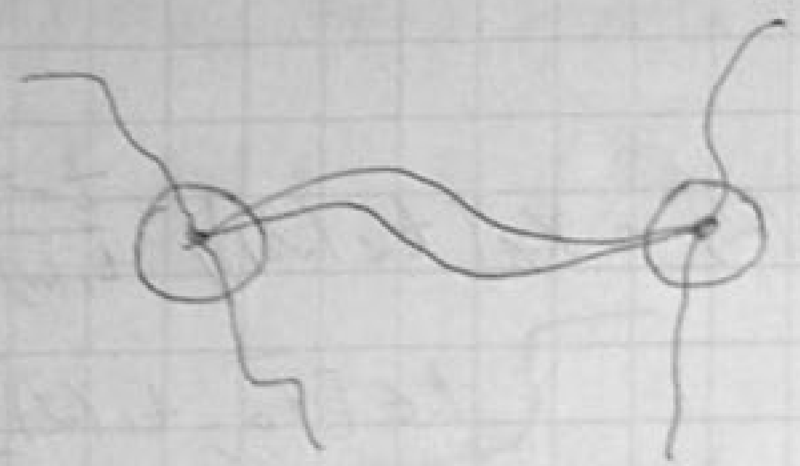
$$- \Big\{f(\tau, x_*(\tau), \dot{x}_*(\tau+0)) - \Big(\frac{\partial f}{\partial \dot{x}}(\tau, x_*(\tau), \dot{x}_*(\tau+0)), \dot{x}_*(\tau+0)\Big)\Big\}\Big]\delta$$

$\forall \delta \in R^1, \forall \Delta \in R^n, \|\Delta\|^2 + \delta^2 \neq 0$

$]\ \Delta = 0$

■ Т. док.

§ Условие трансверсальности с подвижными концами



$$J(x) = \int_{t_0}^{t_1} f(t, x(t), \dot{x}(t))\,dt \to \min$$

На концах $\Phi(t_0, x(t_0)) = 0$, $\Psi(t_1, x(t_1)) = 0$

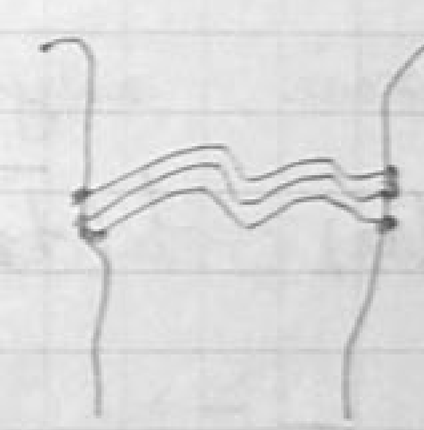
$C^1 \ni \Phi : R^1 \times R^n \to R^k$

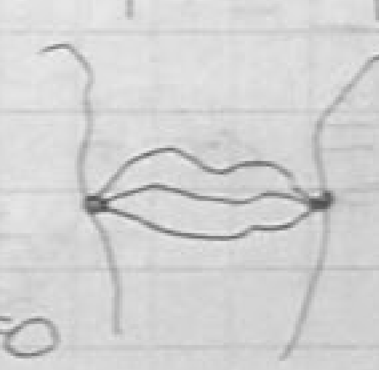
$C^1 \ni \Psi : R^1 \times R^n \to R^l$

t_0, t_1 – могут быть нефикс. величины.

$\exists x_*(t)$, $t \in [\bar{t}_0, \bar{t}_1]$ – решение задачи

$$\Rightarrow \left| \begin{aligned} & J(x) = \int_{\bar{t}_0}^{\bar{t}_1} f(t, x(t), \dot{x}(t))\,dt \to \min \\ & x(\bar{t}_0) = x_*(\bar{t}_0),\ x(\bar{t}_1) = x_*(\bar{t}_1) \end{aligned} \right.$$





$$\Rightarrow \frac{\partial f}{\partial x}(t, x_*(t), \dot{x}_*(t)) - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}}(t, x_*(t), \dot{x}_*(t)) = 0$$

$t \in [\bar{t}_0, \bar{t}_1]$

$\Phi(\bar{t}_0, x_*(\bar{t}_0)) = 0$

$\Psi(\bar{t}_1, x_*(\bar{t}_1)) = 0$ $\Rightarrow$ $k+l$ соотношений

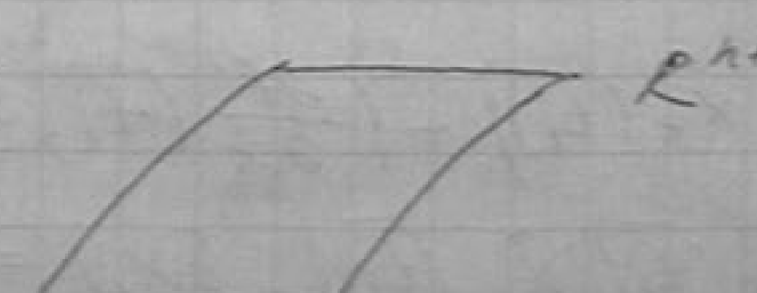
$2n$ const для нахождения x_* $\Rightarrow$ $(2n+2)$ неизв. (нужно)

$(\bar{t}_0, x_*(\bar{t}_0))$, $(\bar{t}_1, x_*(\bar{t}_1))$

$\Psi(t, x) = 0$ l соотн.

↑

задаёт многообразие l мерное



R^{n+1}

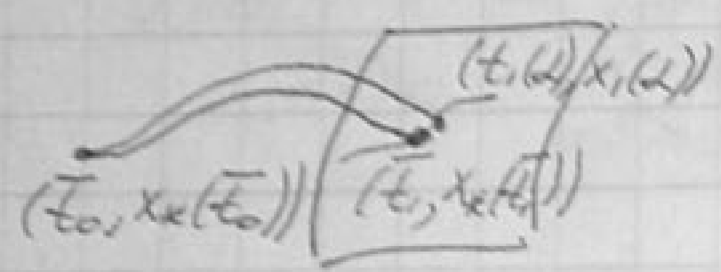
Рассм. касательный вектор $\overset{(\delta t_1, \delta x_1)}{\vee}$ к $\Psi(t,x) = 0$ в $\overset{\text{точке}}{\vee}$ $(\bar{t}_1, x_*(\bar{t}_1))$

$\Rightarrow \exists\, \underset{C^1}{\gamma_1(\alpha)} = (t_1(\alpha), x_1(\alpha))$: $(t_1(0), x_1(0)) = (\bar t_1, x_*(\bar t_1))$
$(\dot t_1(0), \dot x_1(0)) = (\delta t_1, \delta x_1)$

Расм. возмущ. семейство $x^1_\alpha(t) = x_*(t) + (x_1(\alpha) - x_*(t_1(\alpha)))\dfrac{t-\bar t_0}{t_1(\alpha)-\bar t_0}$, $t\in[\bar t_0, t_1(\alpha)]$

Если $\alpha=0$: $x^1_0(t) = x_*(t)$, $t\in[\bar t_0, \bar t_1]$ $\quad \Psi(t,x)=0$

$\alpha\neq 0$: $x^1_\alpha(\bar t_0) = x_*(\bar t_0)$, $x^1_\alpha(t_1(\alpha)) = x_1(\alpha)$



$$\Rightarrow I_1(\alpha) - J(x^1_\alpha) = \int_{t_0}^{t_1(\alpha)} f(t, x^1_\alpha(t), \dot x^1_\alpha(t))\, dt$$

При $\alpha=0$ достигает лок. мин.

$\Rightarrow \dot I_1(0) = 0$

В итоге получится

$$\dot I_1(0) = f(\bar t_1, x_*(\bar t_1), \dot x_*(\bar t_1))\delta t_1 + \int_{t_0}^{\bar t_1} \left(\frac{\partial f}{\partial x}(t, x_*(t), \dot x_*(t)), \delta x_1 - \dot x_*(\bar t_1)\delta t_1\right)\frac{t-\bar t_0}{\bar t_1-\bar t_0}\, dt + \int_{\bar t_0}^{\bar t_1}\left(\frac{\partial f}{\partial \dot x}(t, x_*(t), \dot x_*(t)), \delta x_1 - \dot x_*(\bar t_1)\delta t_1\right)\frac{1}{\bar t_1-\bar t_0}\, dt$$

Интегрируем по частям

$$\dot I_1(0) = f(\bar t_1, x_*(\bar t_1), \dot x_*(\bar t_1))\delta t_1 + \left(\frac{\partial f}{\partial \dot x}(t, x_*(t), \dot x_*(t)), \delta x_1 - \dot x_*(\bar t_1)\delta t_1\right)\cdot \left.\frac{t-\bar t_0}{\bar t_1-\bar t_0}\right|_{\bar t_0}^{\bar t_1} +$$

$$+ \int_{\bar t_0}^{\bar t_1}\left(\cancelto{0}{\frac{\partial f}{\partial x}(t, x_*(t), \dot x_*(t)) - \frac{d}{dt}\frac{\partial f}{\partial \dot x}(t, x_*(t), \dot x_*(t))}, \delta x_1 - \dot x_*(\bar t_1)\delta t_1\right)\cdot\frac{t-\bar t_0}{\bar t_1-\bar t_0}\, dt$$

$$\left\{f(\bar t_1, x_*(\bar t_1), \dot x_*(\bar t_1)) - \left(\frac{\partial f}{\partial \dot x}(\bar t_1, x_*(\bar t_1), \dot x_*(\bar t_1)), \dot x_*(\bar t_1)\right)\right\}\delta t_1 + \left(\frac{\partial f}{\partial \dot x}(\bar t_1, x_*(\bar t_1), \dot x_*(\bar t_1)), \delta x_1\right) = 0$$

для $\forall(\delta t_1, \delta x_1)$ — касательный вектор к $\Psi(t,x)=0$ в точке $(\bar t_1, x_*(\bar t_1))$

↑ Условие трансверсальности

Левый конец $(\bar{t}_0, x_*(\bar{t}_0)) \leftarrow$ в точке $\leftarrow$

$(\delta t_0, \delta x_0)$ – касат вектор к $\Phi(t,x)=0$

$$\exists \underset{C^1}{\underset{\cap}{\jmath_0(\alpha)}} = (t_0(\alpha), x_0(\alpha)): \quad \begin{array}{l}(t_0(0), x_0(0)) = (\bar{t}_0, x_*(\bar{t}_0))\\ (\dot{t}_0(0), \dot{x}_0(0)) = (\delta t_0, \delta x_0)\end{array}$$

$$x_\alpha^0(t) = x_*(t) + (x_0(\alpha) - x_*(t_0(\alpha)))\frac{\bar{t}_1 - t}{\bar{t}_1 - t_0(\alpha)}, \quad t \in [t_0(\alpha), \bar{t}_1]$$

$\alpha = 0 \quad x_0^0(t) = x_*(t), \ t \in [\bar{t}_0, \bar{t}_1]$

$$x_\alpha^0(t_0(\alpha)) = \underset{\substack{\nearrow \\ \text{многообразию } \Phi(t,x)=0}}{x_0(\alpha)}, \quad x_\alpha^0(\bar{t}_1) = x_*(\bar{t}_1)$$

$$I_0(\alpha) = J(x_\alpha^0) = \int_{t_0(\alpha)}^{\bar{t}_1} f(t, x_\alpha^0(t), \dot{x}_\alpha^0(t))\,dt$$

При $\alpha=0$ – лок min

$\dot{I}_0(0) = 0$

(Далее Аналогично предыдущим.)

$$\dot{I}(0) = -f(\bar{t}_0, x_*(\bar{t}_0), \dot{x}_*(\bar{t}_0))\,\delta t_0 + \int_{\bar{t}_0}^{\bar{t}_1}\left(\frac{\partial f}{\partial x}(t, x_*(t), \dot{x}_*(t)), \delta x_0 - \dot{x}_*(\bar{t}_0)\delta t_0\right)\cdot\frac{\bar{t}_1 - t}{\bar{t}_1 - \bar{t}_0}\,dt +$$

$$+ \int_{\bar{t}_0}^{\bar{t}_1}\left(\frac{\partial f}{\partial \dot{x}}(t, x_*(t), \dot{x}_*(t)), \delta x_0 - \dot{x}_*(\bar{t}_0)\delta t_0\right)\frac{1}{\bar{t}_1 - \bar{t}_0}\,dt$$

По частям

$$\dot{I}(0) = -f(\bar{t}_0, x_*(\bar{t}_0), \dot{x}_*(\bar{t}_0))\,\delta t_0 + \left(\frac{\partial f}{\partial \dot{x}}(t, x_*(t), \dot{x}_*(t)), \delta x_0 - \dot{x}_*(\bar{t}_0)\delta t_0\right)\cdot\frac{\bar{t}_1 - t}{\bar{t}_1 - \bar{t}_0}\Bigg|_{\bar{t}_0}^{\bar{t}_1} +$$

$$+ \cancel{\int_{\bar{t}_0}^{\bar{t}_1}\left(\frac{\partial f}{\partial x}(t, x_*(t), \dot{x}_*(t)) - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}}(t, x_*(t), \dot{x}_*(t)), \delta x_0 - \dot{x}_*(\bar{t}_0)\delta t_0\right)\cdot\frac{\bar{t}_1 - t}{\bar{t}_1 - \bar{t}_0}\,dt} \to 0$$

Перегруппировка

$$\left\{f(\bar{t}_0, x_*(\bar{t}_0), \dot{x}_*(\bar{t}_0)) - \left(\frac{\partial f}{\partial \dot{x}}(\bar{t}_0, \bar{t}_0, \bar{t}_0), \dot{x}_*(\bar{t}_0)\right)\right\}\delta t_0 +$$

$+ \left(\dfrac{\partial f}{\partial \dot{x}}(\bar{t}_0, \bar{t}_0, \bar{t}_0), \delta x_0\right) = 0$ для $\forall (\delta t_0, \delta x_0)$ касат

к $\Phi(t,x)=0$ в точке $(\bar{t}_0, x_*(\bar{t}_0))$

16.10.08

$$\begin{cases} \dot{x}(t) = y(t) \\ \frac{d}{dt}\frac{\partial f}{\partial \dot{x}}(t, x(t), y(t)) - \frac{\partial f}{\partial x}(t, x(t), y(t)) = 0 \end{cases}$$

§ Преобразование Лежандра. Ур-ние Гамильтона.

п.1. Преобр. Лежандра

$f(x) \in C^2(\mathbb{R}^n)$

ⓐ $\exists\ \frac{\partial^2 f}{\partial x^2}(x) > 0 \quad \forall x \in \mathbb{R}^n$

ⓑ $\forall p \in \mathbb{R}^n \quad \exists x \in \mathbb{R}^n, \ \frac{\partial f}{\partial x}(x) = p$

Пример $f(x) = \frac{1}{2}(Rx, x)$, $R = R^T > 0$

$$f_*(p) = (p, x) - f(x)\Big|_{x = x(p)}$$ преобразование Лежандра

ⓐ $$\underline{\frac{\partial f_*}{\partial p}(p) = x(p)} + \cancel{\left[\frac{\partial x}{\partial p}(p)\right]^T p - \left[\frac{\partial x}{\partial p}(p)\right]^T \frac{\partial f}{\partial x}(x(p))}$$

$$E = \left[\frac{\partial x}{\partial p}(p)\right]^T \frac{\partial^2 f}{\partial x^2}(x(p)) = \left[\frac{\partial^2 f_*(p)}{\partial p^2}\right]^T \frac{\partial^2 f}{\partial x^2}(x(p))$$

$$\frac{\partial^2 f_*(p)}{\partial p^2} = \frac{\partial x}{\partial p}(p)$$

$$f_{**}(x) = (x, p) - f_*(p)\Big|_{p = p(x)} = f(x)$$

$$\frac{\partial^2 f}{\partial x^2}(x) = R$$

$$\frac{\partial f}{\partial x} = Rx = p \Rightarrow x = R^{-1}p$$

$$f_*(p) = (p, R^{-1}p) - \frac{1}{2}(RR^{-1}p, R^{-1}p) = \frac{1}{2}(R^{-1}p, p)$$

$$\frac{\partial f_*}{\partial p}(p) = R^{-1}p = x \Rightarrow p = Rx$$

$$f_{**}(x) = (x, Rx) - \frac{1}{2}(R^{-1}Rx, Rx) = \frac{1}{2}(Rx, x).$$

п.2. Ур-ние Гамильтона

$\exists\ f(t, x, \dot{x})$

ⓐ $\frac{\partial^2 f}{\partial \dot{x}^2}(t, x, \dot{x}) > 0$ (Усиленное условие Лежандра.)

ⓑ $\forall p \in \mathbb{R}^n \quad \frac{\partial f}{\partial \dot{x}}(t, x, \dot{x}) = p$ разрешимо относит.

$\dot{x} = \dot{x}(t, x, p)$

$\Rightarrow$ Ф-ция Гамильтона $H(t, x, p) = (p, \dot{x}) - f(t, x, \dot{x})\Big|_{\dot{x} = \dot{x}(t,x,p)}$

$\exists\, x = x(t) \in C^2$

$p(t) = \frac{\partial f}{\partial \dot{x}}(t, x(t), \dot{x}(t))$

$\frac{\partial f_x}{\partial p}(p) = x(p)$

(1) $\boxed{\dot{x}(t) = \frac{\partial H}{\partial p}(t, x(t), p(t))}$ $\Leftarrow$ следствие преобр. Лежандра

$\frac{\partial H}{\partial x}(t, x, p) = -\frac{\partial f}{\partial x}(t, x, \dot{x})$

$\frac{\partial H}{\partial x} = \cancel{\left[\frac{\partial \dot{x}}{\partial x}\right]^T p} - \frac{\partial f}{\partial x} - \cancel{\left[\frac{\partial \dot{x}}{\partial x}\right]^T \frac{\partial f}{\partial \dot{x}}} = -\frac{\partial f}{\partial x}$

$\exists\, x(t)$ удовл. ур-нию Эйлера

$\underbrace{\frac{\partial f}{\partial x}}_{-\frac{\partial H}{\partial x}} = \frac{d}{dt}\Big(\underbrace{\frac{\partial f}{\partial \dot{x}}}_{p(t)}\Big)$

$\Rightarrow \boxed{\dot{p}(t) = -\frac{\partial H}{\partial x}(t, x(t), p(t))}$ (2)

(1)+(2) система ур-ний Гамильтона

~ системе ур-ний Эйлера

Лемма

Рассмотрим случай, когда $H(t, x, p) = \underline{\underline{H(x, p)}}$ нет зависимости от t

$\Rightarrow$ Ф-ция $H(x, p)$ сохр. пост. значение вдоль решения системы ур-ний Гамильтона.

Док-во: $\exists\, (\cancel{x}(t), p(t))$ решение системы Гамильтона $\Rightarrow$

$H(x(t), p(t))$

$\frac{d}{dt} H(x(t), p(t)) = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial p}\dot{p} = \frac{\partial H}{\partial x}\cdot\frac{\partial H}{\partial p} + \frac{\partial H}{\partial p}\left(-\frac{\partial H}{\partial x}\right) = 0$

$\exists$ есть (1)+(2)

$\exists\, f(t, x, \dot{x}) = (p, \dot{x}) - H(t, x, p)$

$\dot{x} = \frac{\partial H}{\partial p}(t, x, p)$, $\dot{p} = -\frac{\partial H}{\partial x}(t, x, p)$

$$\frac{d}{dt}\underbrace{\frac{\partial f}{\partial \dot x}}_{p} - \underbrace{\frac{\partial f}{\partial x}}_{-\frac{\partial H}{\partial x}} = \dot p + \frac{\partial H}{\partial x} = 0 \Rightarrow$$ будут выполняться ур-ние Эйлера

Вывод: $\exists x_*(t)$ – решение ур-ние Эйлера $\Rightarrow$

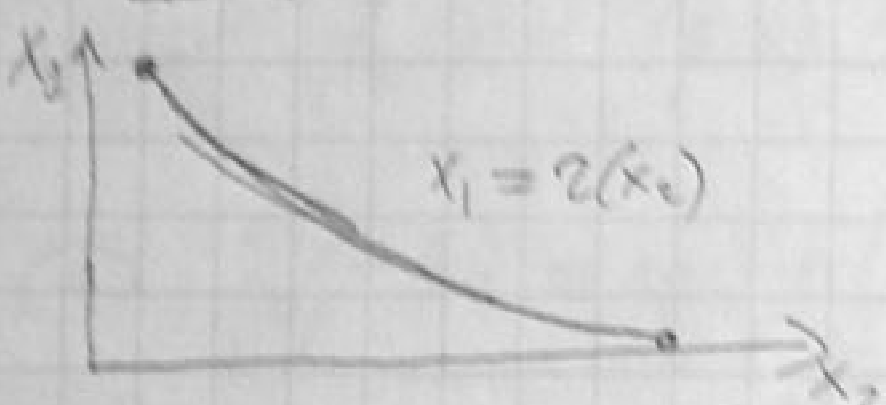
$p_*(t) = \frac{\partial f}{\partial \dot x}(t, x_*(t), \dot x_*(t)) \Rightarrow (x_*(t), p_*(t)) \Rightarrow$

решение системы ур-ний Гамильтона

$\exists (x_*(t), p_*(t))$ – решение системы ур-ний Гамильтона

$f(t, x, \dot x) = (p, \dot x) - H(t, x, p) \Rightarrow x_*(t)$ – реш. ур-ние Эйлера

Пример 1.



$$\begin{cases} J(x) = \int_{t_0}^{t_1} \left[m\frac{\dot x_1^2(t) + \dot x_2^2(t)}{2} - mgx_1(t)\right] dt \to \min \\ g(x_1(t), x_2(t)) = 0, \Rightarrow x_1(t) - r(x_2(t)) = 0 \end{cases}$$

$L(t, x, \dot x, \psi) = m\frac{\dot x_1^2 + \dot x_2^2}{2} - mgx_1 + \psi(x_1 - r(x_2))$

$\frac{\partial L}{\partial \dot x_1} = m\dot x_1 \qquad p(t) = m\dot x(t) \Rightarrow \dot x(t) = \frac{1}{m}p(t)$

$\frac{\partial L}{\partial \dot x_2} = m\dot x_2$

$$H(t, x(t), p(t)) = \left(p(t), \tfrac{1}{m}p(t)\right) + mgx_1 - \psi(t)(x_1(t) - r(x_2(t))) - \frac{m}{2}\frac{p_1^2(t) + p_2^2(t)}{m^2} =$$

$$= \frac{1}{m}\frac{p_1^2(t) + p_2^2(t)}{2} + mgx_1(t) - \psi(t)(x_1(t) - r(x_2(t)))$$

$$H(t, x(t), p(t)) = \underbrace{\frac{p_1^2(t) + p_2^2(t)}{2m} + mgx_1}_{\text{полная механич. энергия системы}}$$

$p(t)$ – импульс.

$\to$ закон сохр. полной мех энергии

Пример №2

$$\begin{cases} J(u)=\frac{1}{2}\int_{t_0}^{t_1}\{(Qy(t),y(t))+(Ru(t),u(t))\}dt\to\min \\ \dot y(t)=Ay(t)+Bu(t),\ y\in R^k,\ u\in R^l \\ y(t_0)=y_0 \end{cases}$$

$Q=Q^T>0$, $R=R^T>0$

$x=(\bar x,\bar{\bar x})$ $\qquad \bar x(t)=y(t)$

$\qquad \bar{\bar x}(t)=\int_{t_0}^{t}u(s)ds$

$x(t_0)=(y_0,0)$

$$\begin{cases} J(x)=\frac{1}{2}\int_{t_0}^{t_1}\{(Q\bar x(t),\bar x(t))+(R\dot{\bar{\bar x}}(t),\dot{\bar{\bar x}}(t))\}dt\to\min \\ \dot{\bar x}(t)-A\bar x(t)-B\dot{\bar{\bar x}}(t)=0 \\ x(t_0)=(y_0,0) \end{cases}$$

$$L(t,x,\dot x,\psi)=\frac{1}{2}(Q\bar x,\bar x)+\frac{1}{2}(R\dot{\bar{\bar x}},\dot{\bar{\bar x}})+(\psi,\dot{\bar x}-A\bar x-B\dot{\bar{\bar x}})$$

$$\frac{\partial L}{\partial \bar x}=Q\bar x-A^T\psi;\quad \frac{\partial L}{\partial \dot{\bar x}}=\psi$$

$$\frac{\partial L}{\partial \bar{\bar x}}=0;\quad \frac{\partial L}{\partial \dot{\bar{\bar x}}}=R\dot{\bar{\bar x}}-B^T\psi$$

$$\frac{\partial^2 L}{\partial \dot{\bar x}^2}=0$$

$$\frac{\partial L}{\partial \dot{\bar x}}=\psi=\bar p$$

если вып. ур-ние Эйлера

$$\frac{\partial L}{\partial \dot{\bar{\bar x}}}=R\dot{\bar{\bar x}}-B^T\psi=\bar{\bar p}=0$$

$$p=(\bar p,\bar{\bar p})$$

$$H(t,x,p,\psi)=(p,\dot x)-\frac{1}{2}(Q\bar x,\bar x)-\frac{1}{2}(R\dot{\bar{\bar x}},\dot{\bar{\bar x}})-(\psi,\dot{\bar x}-A\bar x-B\dot{\bar{\bar x}})$$

$$(\bar p,\dot{\bar x})+(\overset{0}{\bar{\bar p}},\dot{\bar{\bar x}})$$

$$H(t,x(t),p(t),\psi(t))=(\psi(t),A\bar x(t)+B\dot{\bar{\bar x}}(t))-\frac{1}{2}(Q\bar x(t),\bar x(t))-\frac{1}{2}(R\dot{\bar{\bar x}}(t),\dot{\bar{\bar x}}(t))$$

↑ ф-ия Гамильтона

$$\mathcal{H}(t,y,v,\psi)=(\psi,Ay+Bv)-\frac{1}{2}(Qy,y)-\frac{1}{2}(Rv,v)\qquad R>0$$

$$\underbrace{\frac{\partial \mathcal{H}(t,y,v,\psi)}{\partial v} = 0}_{B^T\psi - Rv = 0} \iff \mathcal{H}(t,y,v,\psi) = \max_{u\in R^l} \mathcal{H}(t,y,u,\psi)$$

<u>Теорема</u> Принцип max для лин. квадратич. зад ОУ (ЛКЗОУ)

$\exists\, (x_*(t), u_*(t))$ – опт. решение в ЛКЗОУ

Тогда для $\psi(t)$, удовлетв. зад. Коши:

$$\begin{cases} \dot\psi(t) = -A^T\psi(t) + Qy(t) \\ \psi(t_1) = 0 \end{cases}$$

$$\mathcal{H}(t, y_*(t), u_*(t), \psi(t)) = \max_{v\in R^l} \mathcal{H}(t, y_*(t), v, \psi(t))$$

$$\boxed{u_*(t) = R^{-1}B^T\psi(t)}$$

<u>§ Простейшая задача на сильный min</u>

$$\begin{cases} J(x) = \int_{t_0}^{t_1} f(t, x(t), \dot x(t))\,dt \to \min \\ \qquad\qquad\qquad\qquad (*) \\ x(t_0) = x_0, \quad x(t_1) = x_1 \end{cases}$$

<u>Теорема</u> $\exists\, x_*(t) \in C^1[t_0,t_1]$ доставляет сильный лок min в $(*)$ $\Rightarrow$ для $\forall v \in R^n$

$\Rightarrow E(t, x_*(t), \dot x_*(t), v) \geq 0$, где

$$E(t, x_*, \dot x_*, v) = f(t,x,v) - f(t,x,\dot x) - \left(v - \dot x, \frac{\partial f}{\partial \dot x}(t,x,\dot x)\right)$$

ф-ия Вейерштрасса.

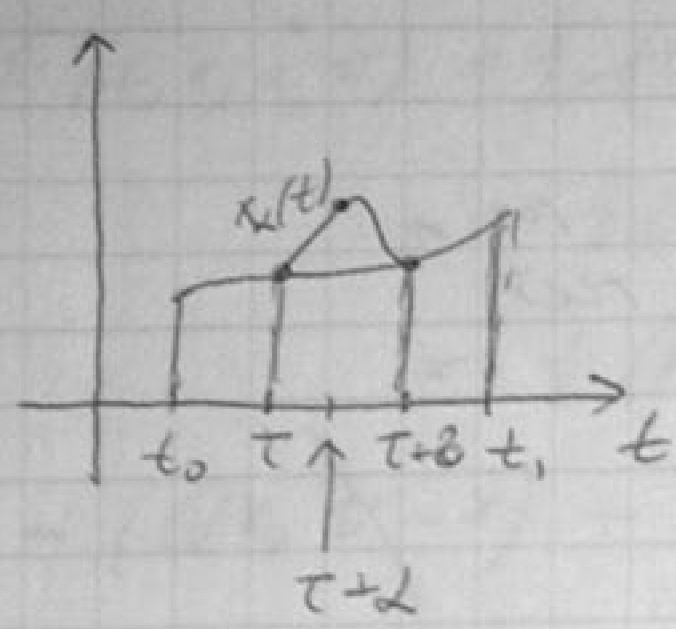
23.10.08

<u>Док-во:</u> Зададим $\tau, \beta > 0, \alpha \geq 0$

$$t_0 \leq \tau \leq t_1, \quad \tau + \beta \leq t_1, \quad \alpha < \beta$$

$\forall v \in R^n$

$$x_\alpha(t) = \begin{cases} x_*(t), & t_0 \leq t < \tau,\ \tau+\beta < t \leq t_1 \\ x_*(\tau) + (t-\tau)v, & \tau \leq t < \tau+\alpha \\ x_*(t) + \frac{\tau+\beta-t}{\beta-\alpha}\left(x_*(\tau) + \alpha v - x_*(\tau+\alpha)\right), & \tau+\alpha \leq t \leq \tau+\beta \end{cases}$$



$$I(\alpha) = J(x_\alpha) = \int_{t_0}^{t_1} f(t, x_\alpha(t), \dot x_\alpha(t))\,dt$$

$\dot I(+0) \geq 0$ необх. условие лок min

$$I(\alpha) = \int_{t_0}^{\tau} f(t, x_*(t), \dot x_*(t))\,dt + \underbrace{\int_{\tau}^{\tau+\alpha} f(t, x_\alpha(t), \dot x_\alpha(t))\,dt}_{I_1(\alpha)} +$$

$$+ \int_{\tau+\alpha}^{\tau+\delta} f(t, x_\alpha(t), \dot x_\alpha(t))\,dt + \int_{\tau+\delta}^{t_1} f(t, x_*(t), \dot x_*(t))\,dt$$

$$I_1(\alpha) = \int_{\tau}^{\tau+\alpha} f(t, x_*(\tau) + (t-\tau)v, v)\,dt$$

$$I_2(\alpha) = \int_{\tau+\alpha}^{\tau+\delta} f\Big(t, x_*(t) + \frac{\tau+\delta-t}{\delta-\alpha}\big(x_*(\tau) + \alpha v - x_*(\tau+\alpha)\big),$$
$$\dot x_*(t) - \frac{1}{\delta-\alpha}\big(x_*(\tau) + \alpha v - x_*(\tau+\alpha)\big)\Big)\,dt$$

$$\dot I_1(+0) = f(\tau, x_*(\tau), v)$$

$$\dot I_2(+0) = -f(\tau, x_*(\tau), \dot x_*(\tau)) + \int_{\tau}^{\tau+\delta} \Big(\frac{\partial f}{\partial x}(t, x_*(t), \dot x_*(t)),\ \frac{v - \dot x_*(\tau)}{\delta}(\tau+\delta-t)\Big)\,dt +$$

$$+ \int_{\tau}^{\tau+\delta} \Big(\frac{\partial f}{\partial \dot x}(t, x_*(t), \dot x_*(t)),\ -\frac{v - \dot x_*(\tau)}{\delta}\Big)\,dt =$$

$$= -f(\tau, x_*(\tau), \dot x_*(\tau)) + \Big(\frac{\partial f}{\partial \dot x}(t, x_*(t), \dot x_*(t)),\ \frac{v - \dot x_*(\tau)}{\delta}(\tau+\delta-t)\Big)\Big|_{\tau}^{\tau+\delta} +$$

$$+ \int_{\tau}^{\tau+\delta} \Big(\cancelto{0}{\frac{\partial f}{\partial x}(t, x_*(t), \dot x_*(t)) - \frac{d}{dt}\frac{\partial f}{\partial \dot x}(t, x_*(t), \dot x_*(t))},\ \frac{v - \dot x_*(\tau)}{\delta}(\tau+\delta-t)\Big)\,dt =$$

$$= -f(\tau, x_*(\tau), \dot x_*(\tau)) - \Big(\frac{\partial f}{\partial \dot x}(\tau, x_*(\tau), \dot x_*(\tau)),\ v - \dot x_*(\tau)\Big)$$

$$\Rightarrow \dot I(+0) = f(\tau, x_*(\tau), v) - f(\tau, x_*(\tau), \dot x_*(\tau)) - \Big(\frac{\partial f}{\partial \dot x}(\tau, x_*(\tau), \dot x_*(\tau)),\ v - \dot x_*(\tau)\Big) \geq 0$$

ф-ия Вейерштрасса

$$\Rightarrow E(\tau, x_*(\tau), \dot x_*(\tau), v) \geq 0$$

Пример 1

$$J(x) = \int_0^1 \dot{x}^3(t)\,dt \longrightarrow \min$$ есть слаб. лок min но сильного нет

$x(0)=0$, $x(1)=1$

$$\frac{d}{dt}\left(3\dot{x}^2\right)=0 \Rightarrow \dot{x}(t)\ddot{x}(t)=0$$

$$\Rightarrow x_*(t)=t$$

$$E(t,x,\dot{x},v) = v^3 - \dot{x}^3 - 3\dot{x}^2(v-\dot{x})$$

$$E(t,x_*(t),\dot{x}_*(t),v) = v^3-1-3(v-1) = (v-1)\left(v^2+v\cancel{+1}-2\right)$$

Если $v=2$ >0
$v=-3$ <0

$\Rightarrow$ сильного min не дает.

Рассм. $\forall$ ф-ию $x(t)\in C^1[0,1]$

$$|\underbrace{\dot{x}(t)-\dot{x}_*(t)}_{\dot{h}(t)}| < 1,\ t\in[0,1], \quad \dot{x}_*(t) = 1$$

$$\eta(t) = x(t)-x_*(t),\quad \eta(0)=\eta(1)=0$$

$$J(x)-J(x_*) = J(x_*+h)-J(x_*) = \int_0^1\left[(1+\dot{h}(t))^3-1\right]dt =$$

$$= \int_0^1\left(3\dot{h}^2(t) + \cancel{3\dot{h}(t)}^{\,0} + \dot{h}^3(t)\right)dt = \int_0^1 \dot{h}^2(t)\underbrace{(3+\dot{h}(t))}_{\vee\, 0}\,dt \geqslant 0$$

Доставляет слаб лок min, но не сильный

Пример №2

$$\begin{cases} J(x) = \frac{1}{2}\int_{t_0}^{t_1}\left\{(Px(t),x(t)) + 2(Qx(t),\dot{x}(t)) + (R\dot{x}(t),\dot{x}(t))\right\}dt \to \min \\ x(t_0)=x_0,\ x(t_1)=x_1 \end{cases}$$

$P=P^T\geqslant 0$, $R=R^T\geqslant 0$, $Q=Q^T$

$$\frac{\partial f}{\partial x} = Px + Q\dot{x},$$

$$\frac{\partial f}{\partial \dot{x}} = Qx + R\dot{x}$$

$$Px + Q\dot{x} - Q\dot{x} - R\ddot{x} = 0$$

$$\boxed{R\ddot{x}(t) - Px(t) = 0}\ \text{УЭ}$$

$$E(t,x,\dot x,v) = \cancel{\tfrac{1}{2}(Px,x)} + (Qx,v) + \tfrac{1}{2}(Rv,v) - \cancel{\tfrac{1}{2}(Px,x)} - (Qx,\dot x) -$$

$$-\tfrac{1}{2}(R\dot x,\dot x) - (Qx + R\dot x,\ v-\dot x) =$$

$$= \tfrac{1}{2}(Rv,v) + \cancel{(Qx, v-\dot x)} - \tfrac{1}{2}(R\dot x,\dot x) -$$

$$- \cancel{(Qx, v-\dot x)} - (R\dot x, v) + (R\dot x,\dot x) =$$

$$= \tfrac{1}{2}(Rv,v) - (R\dot x, v) + \tfrac{1}{2}(R\dot x,\dot x) \underset{\text{т.к. } R-\text{симмет.}}{=} \tfrac{1}{2}\big(R(v-\dot x),(v-\dot x)\big) \geq 0$$

Расcм $\forall x(t) \in C^1[t_0,t_1] \Rightarrow h(t) = x(t) - x_*(t),\ h(t_0) = h(t_1) = 0$

$$J(x) - J(x_*) = J(x_*+h) - J(x_*) = \tfrac{1}{2}\int_{t_0}^{t_1}\{2(Px_*(t),h(t)) +$$

$$+ (Ph(t),h(t)) +$$

$$+ 2(Qx_*(t),\dot h(t)) +$$

$$+ 2(Qh(t),\dot x_*(t)) +$$

$$+ \underline{(Qh(t),\dot h(t))} + \underline{(Q\dot h(t),h(t))} +$$

$$+ 2(R\dot x_*(t),\dot h(t)) + (R\dot h(t),\dot h(t))\}dt =$$

$$\int_{t_0}^{t_1}(Qh(t),\dot h(t))dt = \underbrace{(Qh(t),h(t))\Big|_{t_0}^{t_1}}_{0} - \int_{t_0}^{t_1}(Q\dot h(t),h(t))dt$$

$$\Rightarrow (Qh(t),\dot h(t)) = -(Q\dot h(t),h(t))$$

$$Px = R\ddot x$$

$$\int_{t_0}^{t_1}(Px_*(t),h(t))dt = \int_{t_0}^{t_1}(R\ddot x_*(t),h(t))dt = (R\dot x_*(t),h(t))\Big|_{t_0}^{t_0} - \int_{t_0}^{t_1}(R\dot x_*(t),\dot h(t))dt$$

$$\Rightarrow (Px_*(t),h(t)) = -(R\dot x_*(t),\dot h(t))$$

$$\int_{t_0}^{t_1}(Qx_*(t),\dot h(t))dt = \underbrace{(Qx_*(t),h(t))\Big|_{t_0}^{t_1}}_{0} - \int_{t_0}^{t_1}(Q\dot x_*(t),h(t))dt$$

$$(Qx_*(t),\dot h(t)) = -(Q\dot x_*(t),h(t))$$

$$\Rightarrow \; = \tfrac{1}{2}\int_{t_0}^{t_1}\{\underbrace{(Ph(t),h(t))}_{\geq 0} + \underbrace{(R\dot h(t),\dot h(t))}_{\geq 0}\}dt$$

ВЧ

Пример 3 (линейно-квадратич. задача)

$\exists\, x_*(t)$ – экстремаль

$L(t, x, \dot x, \psi) = \underline{\qquad\qquad}$

выполнено нер-во для ф-ии Вейерштрасса ≥ 0

$\exists\, x(t) \in C^1[t_0, t_1]$

$h(t) = x(t) - x_*(t)$; $t \in [t_0, t_1] \to \bar h(t_0) = 0,\ \bar h(t_0) = 0$

$\dot{\bar h} - A\bar h(t) - B\bar{\bar{\dot h}}(t) = 0$

$$J(x) - J(x_*) = \frac{1}{2}\int_{t_0}^{t_1} \{ \cancel{2(Q\bar x_*(t), \bar h(t))} + (Q\bar h(t), \bar h(t)) + \cancel{2(R\bar{\bar{\dot x}}_*(t), \bar{\bar{\dot h}}(t))} + (R\bar{\bar{\dot h}}(t), \bar{\bar{\dot h}}(t)) \}\, dt \Rightarrow$$

$$\begin{cases} \dot\psi(t) = -A^T\psi(t) + Q\bar x(t), \quad R\bar{\bar{\dot x}}_*(t) - B^T\psi(t) = 0 \ (*) \\ \psi(t_1) = 0 \end{cases}$$

$$\int_{t_0}^{t_1} (Q\bar x_*(t), \bar h(t))\, dt = \int_{t_0}^{t_1} (\dot\psi(t), \bar h(t))\, dt + \int_{t_0}^{t_1} (A^T\psi(t), \bar h(t))\, dt =$$

по частям

$$= \cancel{(\psi(t), \bar h(t))\Big|_{t_0}^{t_1}}^{\,0} - \int_{t_0}^{t_1} (\psi(t), \dot{\bar h}(t))\, dt + \int_{t_0}^{t_1} (\psi(t), A\bar h(t))\, dt$$

$$= -\int_{t_0}^{t_1} (\psi(t), \dot{\bar h}(t) - A\bar h(t))\, dt = -\int_{t_0}^{t_1} (\psi(t), B\bar{\bar{\dot h}})\, dt =$$

$$= -\int_{t_0}^{t_1} (\underbrace{B^T\psi(t)}_{\overset{(*)}{=}\, R\bar{\bar{\dot x}}_*(t)}, \bar{\bar{\dot h}}(t))\, dt$$

$\Rightarrow\ \geq 0$

$\Rightarrow$ экстремаль – сильный min

Пример 4 (задача о простом движении)

$$J(x) = \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt \to \min$$

$\dot x(t) = u(t)$

$x(t_0) = x_0$

$\psi(t) = \dfrac{\partial f}{\partial \dot x}(t, x_*(t), \dot x_*(t))$ (**)

$$\begin{cases} J(x) = \int_{t_0}^{t_1} f(t, x(t), \dot{x}(t))\, dt \\ x(t_0) = x_0, \quad x(t_1) = x_1 \end{cases}$$

$$f(t, x_*(t), v) - f(t, x_*(t), \dot{x}_*(t)) - \left(\frac{\partial f}{\partial \dot{x}}(t, x_*(t), \dot{x}_*(t)), v - \overset{= u_*(t)}{\dot{x}_*(t)}\right) \geq 0$$

необх условие сильного лок min

<u>Теорема</u>

$\exists\, u_*(t)$ — оптим. упр-ние и $x_*(t)$ — опт. траектория для некот. ф-ии $\psi(t)$ (**)

$$-f(t, x_*(t), u_*(t)) + (\psi(t), u_*(t)) \geq -f(t, X_*(t), v) + (\psi(t), v) \quad \forall v \in \mathbb{R}^n$$

или

$$-f(t, x_*(t), u_*(t)) + (\psi(t), u_*(t)) = \sup_{v \in \mathbb{R}} \left[-f(t, x_*(t), v) + (\psi(t), v)\right] \quad \forall v \in \mathbb{R}^n$$

$$\| \quad u_*$$

30.10.08

<u>§ Поле ~~экстр~~ экстремалей</u>. <u>Ур-ние Гамильтона — Якоби</u>

$$\begin{cases} J(x) = \int_{t_0}^{t_1} f(t, x(t), \dot{x}(t))\, dt \to \min \\ x(t_0) = x_0, \quad x(t_1) = x_1 \end{cases}$$

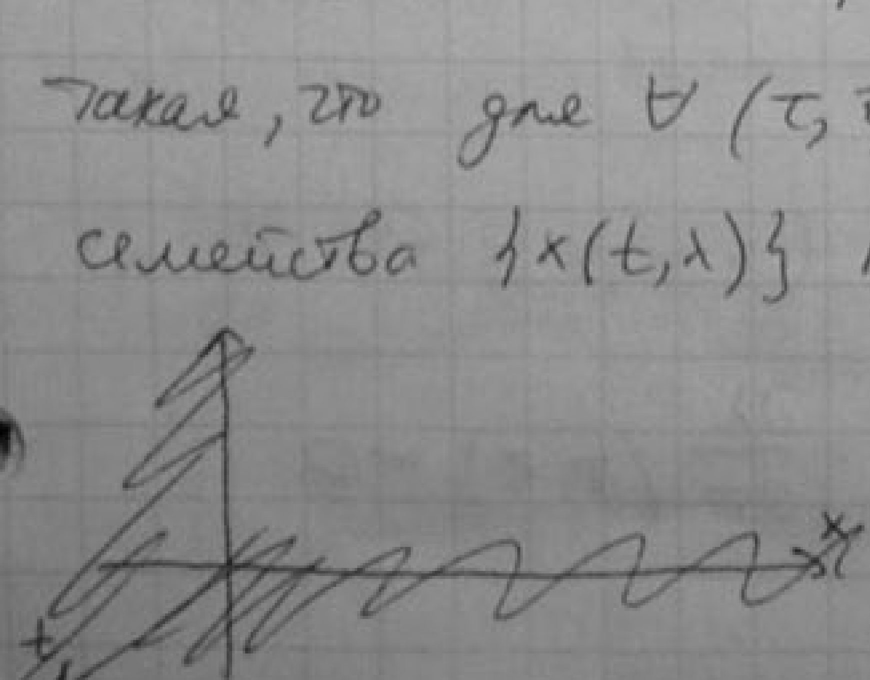
$\exists\, x_*(t),\ t \in [t_0, t_1]$ — экстремаль

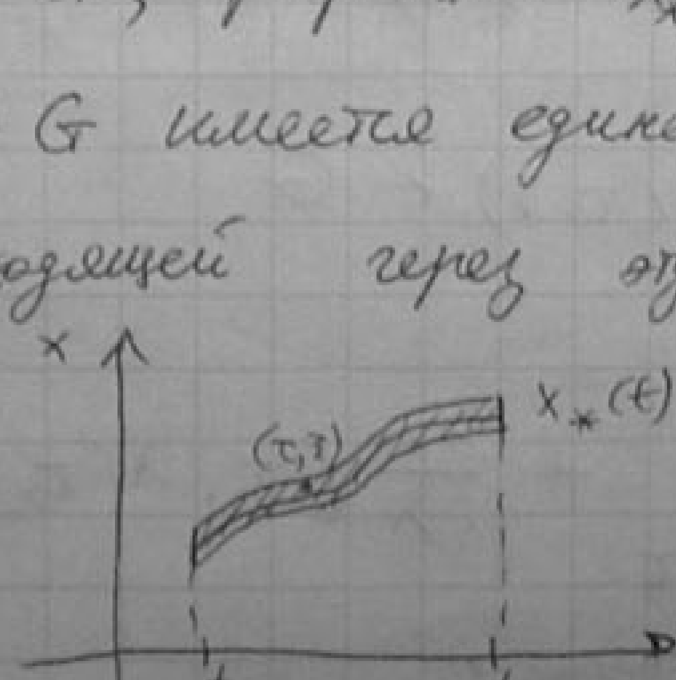
$\exists\, X_*(t) \in \{x(t, \lambda)\}$, $x(t, \lambda) \in C^1([t_0, t_1] \times \mathbb{R}^n)$

$\lambda \in \underset{\text{открытое мн-во}}{\Lambda} \subset \mathbb{R}^n$

$x_*(t) = x(t, 0)$

<u>Опр 1</u> Говорим, что экстремаль окружена полем экстремалей $\{x(t, \lambda)\}$, если $\exists$ окр-ть G, графика $\Gamma_{x_*} = \{(t, x_*(t)) \in \mathbb{R}^{n+1};\ t \in [t_0, t_1]\}$ такая, что для $\forall\, (\tau, z) \in G$ имеется единственная экстремаль из семейства $\{x(t, \lambda)\}$ проходящей через эту точку





$\exists$ ф-ция $\lambda : G \to \mathbb{R}^n$, $\lambda = \lambda(\tau, \xi) \in C^1(G)$, что

$x(\tau, \lambda) = \xi \Leftrightarrow \lambda = \lambda(\tau, \xi)$

<u>Опр</u>$_2$ Ф-ция $u : G \to \mathbb{R}^n$, $u(\tau, \xi) = \frac{dx}{dt}(t, \lambda(\tau, \xi))\Big|_{t=\tau}$

~~угол~~ ф-ция наклона поля

<u>Опр</u>$_3$ Пусть $\exists\ (\hat{t}, \hat{x})$, что $x(\hat{t}, \lambda) = \hat{x}$ для всех $\lambda \in \Lambda$

Тогда говорят, что экстремаль $x_*(t)$ окружена центральным полем экстремалей, $(\hat{t}, \hat{x})$ — центр поля

$\{x(t, \lambda)\}$ — центральное поле экстремалей

<u>Предположение</u>

Пусть $\{x(t, \lambda)\}$ — центральное поле экстремалей

$x(t, \lambda) \in C^2([t_0, t_1] \times \Lambda)$ с центром $(\hat{t}, \hat{x})$, окружающая $x_*(t)$

Пусть $f(t, x, \dot{x}) \in C^2(\Gamma_{x_* \dot{x}_*})$

$\Gamma_{x_* \dot{x}_*} = \{(t, x_*(t), \dot{x}_*(t)) \in \mathbb{R}^{2n+1} : t \in [t_0, t_1]\}$

$S(\tau, \xi) = \int\limits_{\hat{t}}^{\tau} f(t, x(t, \lambda(\tau, \xi)), \dot{x}(t, \lambda(\tau, \xi)))\, dt$ — ф-ция действие центрального поля экстремалей $\{x(t, \lambda)\}$

<u>Теорема</u> Пусть экстремаль $x_*(t)$ окружена центральным полем экстремалей $\{x(t, \lambda)\}$ с центром $(\hat{t}, \hat{x})$ и определена ф-ция действие $S(\tau, \xi) \in C^1(G)$. Тогда её частные производные имеют вид:

$$\frac{\partial S}{\partial \tau}(\tau, \xi) = f(\tau, \xi, u(\tau, \xi)) - \left(\frac{\partial f}{\partial \dot{x}}(\tau, \xi, u(\tau, \xi)), u(\tau, \xi)\right)$$

$$\frac{\partial S}{\partial \xi}(\tau, \xi) = \frac{\partial f}{\partial \dot{x}}(\tau, \xi, u(\tau, \xi))$$

$x(\tau, \lambda(\tau, \xi)) = \xi \quad \forall (\tau, \xi) \in G$

дифф по τ

$$\Rightarrow \underbrace{\frac{dx}{dt}(\tau, \lambda(\tau, \xi))\Big|_{t=\tau}}_{u(\tau, \xi) \text{ — наклон поля}} + \frac{\partial x}{\partial \lambda}(\tau, \lambda(\tau, \xi)) \cdot \frac{\partial \lambda}{\partial \tau}(\tau, \xi) = 0$$

$$\Rightarrow \boxed{u(\tau,\xi) = -\frac{\partial x}{\partial\lambda}(\tau,\lambda(\tau,\xi))\,\frac{\partial\lambda}{\partial\tau}(\tau,\xi)}$$ выр-ние для наклона поле

$$\frac{\partial x}{\partial\lambda}(\tau,\lambda(\tau,\xi))\,\frac{\partial\lambda}{\partial\xi}(\tau,\xi) = E$$

$$x(\hat t,\lambda) = \hat x \quad \forall\lambda\in\Lambda \Rightarrow x(\hat t,\lambda(\tau,\xi)) = \hat x$$

$$\frac{\partial S}{\partial\tau}(\tau,\xi) = f(\tau,\underbrace{x(\tau,\overset{\lambda(\tau,\xi)}{\lambda})}_{\xi},\overbrace{\frac{dx}{dt}(t,\lambda(\tau,\xi))\Big|_{t=\tau}}^{u(\tau,\xi)}) +$$

$$+\int_{\hat t}^{\tau}\Big\{\Big(\frac{\partial f}{\partial x}(t,x(t,\lambda(\tau,\xi)),\dot x(t,\lambda(\tau,\xi))),\ \frac{\partial x}{\partial\lambda}(t,\lambda(\tau,\xi))\cdot\frac{\partial\lambda}{\partial\tau}(\tau,\xi)\Big) +$$

$$+\Big(\frac{\partial f}{\partial\dot x}(t,x(t,\lambda(\tau,\xi)),\dot x(t,\lambda(\tau,\xi))),\ \frac{\partial\dot x}{\partial\lambda}(t,\lambda(\tau,\xi))\frac{\partial\lambda}{\partial\tau}(\tau,\xi)\Big)\Big\}dt =$$

$$= f(\tau,\xi,u(\tau,\xi)) + \Big(\frac{\partial f}{\partial\dot x}(t,x(t,\lambda(\tau,\xi)),\dot x(t,\lambda(\tau,\xi))),\ \frac{\partial x}{\partial\lambda}(t,\lambda(\tau,\xi))\cdot\frac{\partial\lambda}{\partial\tau}(\tau,\xi)\Big)\Big|_{\hat t}^{\tau} +$$

$$+\int_{\hat t}^{\tau}\Big(\underbrace{\frac{\partial f}{\partial x}(t,x(t,\lambda(\tau,\xi)),\dot x(t,\lambda(\tau,\xi))) - \frac{d}{dt}\frac{\partial f}{\partial\dot x}(t,x(t,\lambda(\tau,\xi)),\dot x(t,\lambda(\tau,\xi)))}_{0},\ \frac{\partial x}{\partial\lambda}(t,\lambda(t,\xi))\frac{\partial\lambda}{\partial\tau}(\tau,\xi)\Big)dt$$

$$\Rightarrow \frac{\partial S}{\partial\tau}(\tau,\xi) = f(\tau,\xi,u(\tau,\xi)) + \Big(\frac{\partial f}{\partial\dot x}(\tau,x(\tau,\lambda(\tau,\xi))),\underbrace{\frac{dx}{dt}(t,\lambda(\tau,\xi))\Big|_{t=\tau}}_{u(\tau,\xi)},\ \underbrace{\frac{\partial x}{\partial\lambda}(\tau,\lambda(\tau,\xi))\frac{\partial\lambda}{\partial\tau}(\tau,\xi)}_{-u(\tau,\xi)}\Big),$$

$$\Rightarrow \frac{\partial S}{\partial\xi}(\tau,\xi) = \int_{\hat t}^{\tau}\Big\{\frac{\partial x}{\partial\lambda}(t,\lambda(\tau,\xi))\frac{\partial\lambda}{\partial\xi}(\tau,\xi)\frac{\partial f}{\partial x}(t,x(t,\lambda(t,\xi)),\dot x(t,\lambda(\tau,\xi))) +$$

$$+\frac{\partial\dot x}{\partial\lambda}(t,\lambda(\tau,\xi))\frac{\partial\lambda}{\partial\xi}(\tau,\xi)\frac{\partial f}{\partial\dot x}(t,x(t,\lambda(t,\xi)),\dot x(t,\lambda(\tau,\xi)))\Big\}dt$$

по частям

$$= \frac{\partial x}{\partial\lambda}(t,\lambda(\tau,\xi))\frac{\partial\lambda}{\partial\xi}(\tau,\xi)\frac{\partial f}{\partial\dot x}(t,x(t,\lambda(\tau,\xi)),\dot x(t,\lambda(\tau,\xi)))\Big|_{\cancel{\hat t}\to 0}^{\tau} +$$

$$+\int_{\hat t}^{\tau}\frac{\partial x}{\partial\lambda}(t,\lambda(\tau,\xi))\frac{\partial\lambda}{\partial\xi}(\tau,\xi)\underbrace{\Big(\frac{\partial f}{\partial x}(t,x(t,\lambda(\tau,\xi)),\dot x(t,\lambda(\tau,\xi))) - \frac{d}{dt}\frac{\partial f}{\partial\dot x}(t,x(t,\lambda(\tau,\xi)),\dot x(-\text{"}-))\Big)}_{0}dt$$

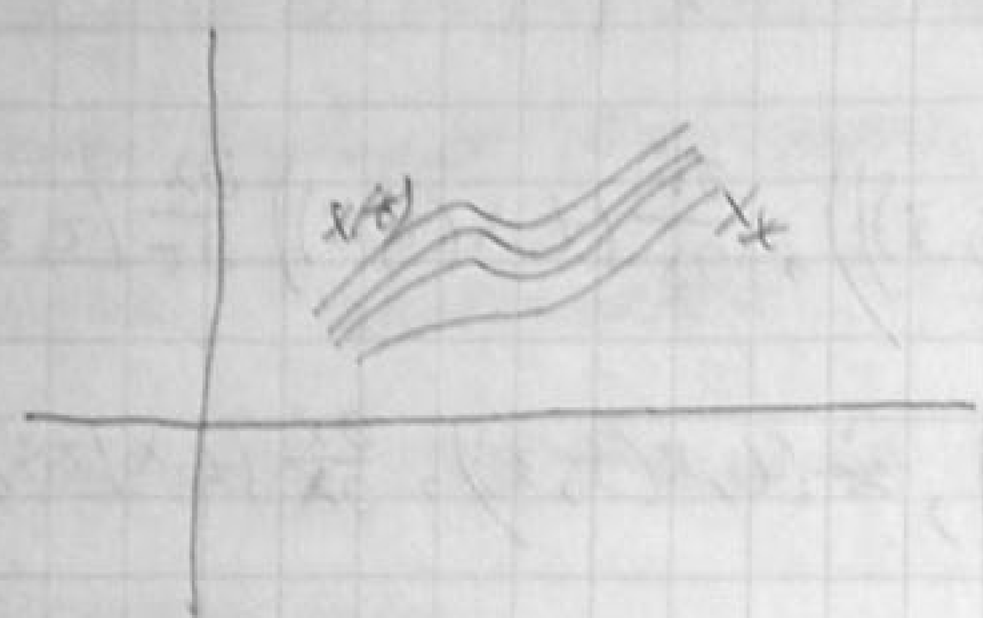
$$\Rightarrow \frac{\partial S}{\partial\xi}(\tau,\xi) = \underbrace{\frac{\partial x}{\partial\lambda}(\tau,\lambda(\tau,\xi))\frac{\partial\lambda}{\partial\xi}(\tau,\xi)}_{E}\frac{\partial f}{\partial\dot x}(\tau,\underbrace{x(\tau,\lambda(\tau,\xi))}_{\xi},\underbrace{\frac{dx}{dt}(t,\lambda(\tau,\xi))\Big|_{t=\tau}}_{u(\tau,\xi)})$$

$$-\frac{\partial f}{\partial\dot x}(\tau,\xi,u(\tau,\xi))$$

$$dS(\tau,\xi) = \frac{\partial S}{\partial \tau}(\tau,\xi)\,d\tau + \left(\frac{\partial S}{\partial \xi}(\tau,\xi), d\xi\right) =$$
$$= f(\tau,\xi,u(\tau,\xi))\,d\tau + \left(\frac{\partial f}{\partial \dot x}(\tau,\xi,u(\tau,\xi)), d\xi - u(\tau,\xi)\,d\tau\right)$$

$\forall x(t): (t, x(t)) \in G$
кус-диф

$$dS(t,x(t)) = \left\{ f(t,x(t),u(t,x(t))) + \left(\frac{\partial f}{\partial \dot x}(t,x(t),u(t,x(t))), \dot x(t) - u(t,x(t))\right)\right\} dt$$

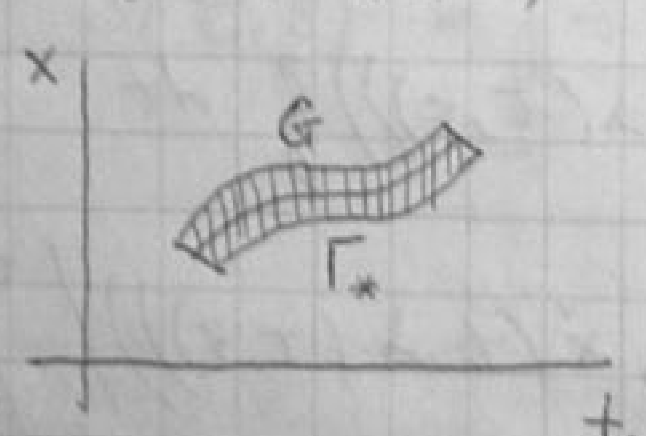


$$u(t,x_*(t)) = \dot x_*(t)$$

$$\boxed{dS(t,x_*(t)) = f(t,x_*(t),\dot x_*(t))\,dt}$$

6.11.08

Теорема] при сделанных предположениях экстремаль $x_*(t)$ окружена центральным полем экстремалей $\{x(t,\alpha)\}$, покрыв. некот. окр. G графика Γ_{x_*}, пусть $\forall(\tau,\xi)\in G$ и $\forall v \in \mathbb{R}^n$ $E(\tau,\xi,u(\tau,\xi),v) \geq 0$. Тогда $x_*(t)$ доставляет сильный лок. min



Док-во: $\forall x(t)$ – кусочно-дифф, удовл. краевым условием + расположена в G

$$J(x_*) = \int_{t_0}^{t_1} f(t,x_*(t),\dot x_*(t))\,dt = \int_{t_0}^{t_1} dS(t,x_*(t)) = S(t_1,x_*(t_1)) - S(t_0,x_*(t_0)) =$$
$$= S(t_1,x_1) - S(t_0,x_0) = S(t_1,x(t_1)) - S(t_0,x(t_0)) = \int_{t_0}^{t_1} dS(t,x(t))$$
$$J(x) - J(x_*) = \int_{t_0}^{t_1} f(t,x(t),\dot x(t))\,dt - \int_{t_0}^{t_1} dS(t,x(t)) =$$

$f(t,x(t),u(t,x(t))) + \left(\frac{\partial f}{\partial \dot x}(t,x(t),u(t,x(t))), \dot x(t) - u(t,x(t))\right)$

$$= \int_{t_0}^{t_1} \left\{ f(t,x(t),\dot x(t)) - f(t,x(t),u(t,x(t))) - \left(\frac{\partial f}{\partial \dot x}(t,x(t),u(t,x(t))), \dot x(t) - u(t,x(t))\right)\right\} dt \ominus$$

$$\textcircled{=} \int_{t_0}^{t_1} E(t, x(t), u(t, x(t)), \dot{x}(t))\,dt \geqslant 0$$

$$E(t, x, \dot{x}, v) \overset{def}{=} f(t, x, v) - f(t, x, \dot{x}) - \left(\frac{\partial f}{\partial \dot{x}}(t, x, \dot{x}), v - \dot{x}\right)$$

x_* доставляет сильный лок min

Т. док

Уравнение Гамильтона – Якоби

$$\frac{\partial^2 f}{\partial \dot{x}^2}(t, x_*(t), \dot{x}_*(t)) > 0, \quad t \in [t_0, t_1]$$

В окр-ти G графика Γ_{x_*} выполним преобраз. Лежандра

$$\frac{\partial f}{\partial \dot{x}}\left(\tau, \xi, \frac{dx}{dt}(t, \lambda(\tau, \xi))\Big|_{t=\tau}\right) = p(\tau, \xi)$$

ф-ия Гамильтона

$$H(\tau, \xi, p(\tau, \xi)) = \left(p(\tau, \xi), \frac{dx}{dt}(t, \lambda(\tau, \xi))\Big|_{t=\tau}\right) - f\left(\tau, \xi, \frac{dx}{dt}(t, \lambda(\tau, \xi))\Big|_{t=\tau}\right)$$

$$H\Big(\tau, \xi, \underbrace{\frac{\partial S}{\partial \xi}(\tau, \xi)}_{\text{вместо } p(\tau,\xi)}\Big) = \Big(\underbrace{\frac{\partial S}{\partial \xi}(\tau, \xi)}_{= \frac{\partial f}{\partial \dot{x}}(\tau, \xi, u(\tau, \xi))}, u(\tau, \xi)\Big) - f(\tau, \xi, u(\tau, \xi)) \textcircled{=}$$

$$\textcircled{=} -\left\{f(\tau, \xi, u(\tau, \xi)) - \left(\frac{\partial f}{\partial \dot{x}}(\tau, \xi, u(\tau, \xi)), u(\tau, \xi)\right)\right\} = -\frac{\partial S}{\partial \tau}(\tau, \xi)$$

$$\begin{cases} \frac{\partial S}{\partial \tau}(\tau, \xi) + H\left(\tau, \xi, \frac{\partial S}{\partial \xi}(\tau, \xi)\right) = 0 \\ S(\hat{t}, \hat{x}) = 0 \end{cases}$$

Ур-ние Гамильтона – Якоби (Г – Я)

$\{x(t, \lambda)\}$

Рассм. G графика Γ_{x_*}

$\exists$ в G задана кривая γ

$$\Rightarrow \Gamma(\gamma) = \int_\gamma \{(p(\tau, x), dx) - H(\tau, x, p(\tau, x))\,d\tau\}$$

$\Gamma(\gamma)$ — интеграл Гильберта

Лемма 1 (об инвариантности инт-ала Гильберта)

$\exists$ в G $\exists S(\tau, x) \in C^1(G)$ – решение ур-ние Г-Я

При $p(\tau, x) = \frac{\partial S}{\partial x}(\tau, x)$ значение $\Gamma(\gamma)$ по параметрицов. кривой γ зависит лишь от концов

Док-во: есть γ, $\tau = t$, $x = x(t)$, $t \in [t_0, t_1]$

$$\Gamma(\gamma) = \int_{t_0}^{t_1} \left\{ \frac{\partial S}{\partial x}(t, x(t)), \dot{x}(t)) + \frac{\partial S}{\partial t}(t, x(t)) \right\} dt =$$

$$= \int_{t_0}^{t_1} dS(t, x(t)) = S(t_1, x(t_1)) - S(t_0, x(t_0)) = S(t_1, x_1) - S(t_0, x_0)$$

Лемма док

Лемма 2 (О существовании поля)

$\exists S(t, x) \in C^2(G)$ – решение ур-ние Г-Я в обл-ти G, тогда в окр-ти G графика Γ_{x_*} $\exists$ поле экстремалей

Док-во: Рассм $\forall (\tau, \xi) \in G$ (точку), $\exists p(t, x) = \frac{\partial S}{\partial x}(t, x)$

зад. Коши

$$\exists \begin{cases} \dot{x}(t) = \frac{\partial H}{\partial p}(t, x(t), p(t, x(t))) \\ x(\tau) = \xi \end{cases}$$

$$\dot{p}(t) = \frac{d}{dt}\left(\frac{\partial S}{\partial x}(t, x(t))\right) = \frac{\partial^2 S}{\partial t \partial x}(t, x(t)) + \frac{\partial^2 x}{\partial x^2}(t, x(t))\dot{x}(t) =$$

$$= \frac{\partial}{\partial x}\Big(\underbrace{\frac{\partial S}{\partial t}(t, x(t))}_{= -H(t, x(t), \frac{\partial S}{\partial x}(t, x(t)))}\Big) + \frac{\partial^2 S}{\partial x^2}(t, x(t))\,\dot{x}(t) =$$

$$= -\frac{\partial}{\partial x}\left(H(t, x(t), p(t, x(t)))\right) + \frac{\partial^2 S}{\partial x^2}(t, x(t))\,\dot{x}(t) =$$

$$= -\frac{\partial H}{\partial x}(t, x(t), p(t, x(t))) - \cancel{\frac{\partial p}{\partial x}(t, x(t))\frac{\partial H}{\partial p}(t, x(t), p(t, x(t)))} + \cancel{\frac{\partial^2 S}{\partial x^2}(t, x(t))}\underbrace{\dot{x}(t)}_{= \frac{\partial H}{\partial p}}$$

$$\Rightarrow \dot{p}(t) = -\frac{\partial H}{\partial x}(t, x(t), p(t, x(t)))$$ $\leftarrow$ 2-е ур-ние системы ур-ний Гамильтона (эквив. форма записи ур-нию Эйлера)

Лемма 3 $\exists\ \gamma_* = \{ (t, x_*(t)) \in R^{n+1},\ t \in [t_0, t_1] \} = \Gamma_{x_*}$

экстремаль, которая окруж. полем экстремалей, ~~пор~~ порож $S(t,x) \in C^1(G)$ – решение ур-ния Г-Я

Тогда $\Gamma(\gamma_*) = \int_{t_0}^{t_1} f(t, x_*(t), \dot{x}_*(t))\,dt = J(x_*)$

Док-во:

$$\Gamma(\gamma_*) = \int_{t_0}^{t_1} \{ p(t, x_*(t)), \dot{x}_*(t)) - H(t, x_*(t), p(t, x_*(t)))\}\,dt =$$

$$\overset{\text{по опр. } H}{=} \int_{t_0}^{t_1} f(t, x_*(t), \dot{x}_*(t))\,dt = J(x_*)$$

Теорема $\exists$ экстремаль $x_*(t)$ окружена полем экстремалей, которое покрывает окр-ть G графика Γ_{x_*} и порождена $S(t,x) \in C^1(G)$ – решение ур-ния Г-Я

$\exists$ в G $E(t, x, \frac{\partial H}{\partial p}(t,x)), v) \geq 0$, где

$p(t,x) = \frac{\partial S}{\partial x}(t,x)$ и $\forall v \in R^n$

Тогда $x_*(t)$ доставляет сильный лок. min

Док-во: $\exists\ \forall$ кусочно-дифф $x(t)$, которая целиком лежит в G:

$(t, \overset{\gamma}{x(t)}) \in G\ \forall t \in [t_0, t_1]$

$$J(x) - J(x_*) = \int_{t_0}^{t_1} f(t, x(t), \dot{x}(t))\,dt - \int_{t_0}^{t_1} f(t, x_*(t), \dot{x}_*(t))\,dt -$$

$$= \int_{t_0}^{t_1} f(t, x(t), \dot{x}(t)) - \int_{\gamma_*} \{ \underbrace{(p(t,x), dx) - H(t, x, p(t,x))\,dt}_{I(\gamma_*) = \Gamma(\gamma)} \} =$$

$$= \int_{t_0}^{t_1} f(t, x(t), \dot{x}(t))dt - \int_{\gamma} \{ (p(t,x), dx) - H(t, x, p(t,x))\,dt \} =$$

$$= \int_{t_0}^{t_1} \{ f(t, x(t), \dot{x}(t))dt - (p(t, x(t)), \dot{x}(t)) + H(t, x(t), p(t, x(t))) \}\,dt =$$

$$\Rightarrow p(t, x(t)) = \frac{\partial f}{\partial \dot{x}}(t, x(t), \frac{\partial H}{\partial p}(t, x(t), p(t, x(t))))$$

$$(p(t, x(t)), \dot{x}(t)) = \left(\frac{\partial f}{\partial \dot{x}}(t, x(t), \frac{\partial H}{\partial p}(t, x(t), p(t, x(t)))), \dot{x}(t) \right)$$

$$H(t, x(t), p(t, x(t))) = \left(\frac{\partial f}{\partial \dot{x}}(t, x(t), \frac{\partial H}{\partial p}(t, x(t), p(t, x(t)))), \frac{\partial H}{\partial p}(t, x(t), p(t, x(t))) \right) - f(t, x(t), \frac{\partial H}{\partial p}(t, x(t), p(t, x(t))))$$

$$= \int_{t_0}^{t_1} \{ f(t, x(t), \dot{x}(t)) - f(t, x(t), \frac{\partial H}{\partial p}(t, x(t), p(t, x(t)))) -$$

$$- (\frac{\partial f}{\partial \dot{x}}(t, x(t), \frac{\partial H}{\partial p}(t, x(t), p(t, x(t)))), \dot{x}(t) - \frac{\partial H}{\partial p}(t, x(t), p(t, x(t)))) \} dt =$$

$$= \int_{t_0}^{t_1} E(t, x(t), \frac{\partial H}{\partial p}(t, x(t), p(t, x(t))), \dot{x}(t)) dt \geqslant 0$$

ч.т.д.

Пример 1 Задача Коши для ур-ния Г-Я

Квадратич. задача ВИ

$$f(t, x, \dot{x}) = \frac{1}{2}(\underset{P^T \geqslant 0}{\underset{\|}{P}}x, x) + (\underset{Q^T}{\underset{\|}{Q}}x, \dot{x}) + \frac{1}{2}(\underset{R^T > 0}{\underset{\|}{R}}\dot{x}, \dot{x})$$

$$\frac{\partial f}{\partial \dot{x}} = Qx + R\dot{x} = p \Rightarrow \dot{x} = R^{-1}(p - Qx)$$

$$\Rightarrow H(t, x, p) = (p, R^{-1}(p - Qx)) - \frac{1}{2}(Px, x) - (Qx, R^{-1}(p - Qx))$$

$$-\frac{1}{2}(RR^{-1}(p - Qx), R^{-1}(p - Qx)) =$$

$$= \frac{1}{2}(p - Qx, R^{-1}(p - Qx)) - \frac{1}{2}(Px, x)$$

$$\Rightarrow H(t, x, p) = \frac{1}{2}(p - Qx, R^{-1}(p - Qx)) - \frac{1}{2}(Px, x)$$

$$\frac{\partial S}{\partial t} + H(t, x, \frac{\partial S}{\partial x}(t, x)) = 0$$

$$S(t, x) = \frac{1}{2}(F(t)x, x), \quad F(t) = F^T(t)$$

$$S(\hat{t}, \hat{x}) = 0, \quad \boxed{F(\hat{t}) = 0}$$

$$\frac{\partial S}{\partial x}(t, x) = F(t)x$$

$$\frac{\partial S}{\partial t} = \frac{1}{2}(\dot{F}(t)x, x)$$

$$(\{\dot{F}(t) + (F(t) - Q)R^{-1}(F(t) - Q) - P\}x, x) = 0$$

$$\Rightarrow \begin{cases} \dot{F}(t) + (F(t) - Q)R^{-1}(F(t) - Q) - P = 0 \\ F(\hat{t}) = 0 \end{cases}$$

Дифф ур-ние Риккати

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§ Условие второго порядка в задачах ВИ 13.11.08

$$\begin{cases} J(x) = \int_{t_0}^{t_1} f(t, x(t), \dot{x}(t))\, dt \to \min \\ x(t_0) = x_0, \; x(t_1) = x_1 \end{cases}$$

$f(t, x, \dot{x}) \in C^2_{x\dot{x}}$ кусочно дифф. ф-ция

<u>Теорема</u> $\exists\, x_*(t)$ доставляет слаб лок min. Тогда в точках непрерывности $\dot{x}_*(t)$: $\frac{\partial^2 f}{\partial \dot{x}^2}(t, x_*(t), \dot{x}_*(t)) \geq 0 \quad \forall t \in [t_0, t_1]$

<u>Док-во:</u> $\forall \eta(t)$ – кусочно-дифф. $\eta(t)$: $\eta(t_0) = 0$, $\eta(t_1) = 0$

$$x_\alpha(t) = x_*(t) + \alpha \eta(t), \quad \dot{x}_\alpha(t) = \dot{x}_*(t) + \alpha \dot{\eta}(t)$$

$$I(\alpha) = J(x_\alpha) = \int_{t_0}^{t_1} f(t, x_\alpha(t), \dot{x}_\alpha(t))\, dt$$

$\dot{I}(0) = 0$, $\ddot{I}(0) \geq 0$ $\quad |\alpha| \max\{\max_{[t_0,t_1]} \|\eta(t)\|, \max \|\dot{\eta}(t)\|\} < \varepsilon$

$$\ddot{I}(\alpha)\Big|_{\alpha=0} = \int_{t_0}^{t_1} \Big\{ \Big(\underbrace{\frac{\partial^2 f}{\partial x^2}(t, x_*(t), \dot{x}_*(t))}_{P(t)} \eta(t), \eta(t) \Big) +$$

$$+ \Big(\frac{\partial^2 f}{\partial \dot{x} \partial x}(t, x_*(t), \dot{x}_*(t)) \eta(t), \dot{\eta}(t) \Big) \; (*) + \Big(\frac{\partial^2 f}{\partial x \partial \dot{x}}(-\text{"}-) \dot{\eta}(t), \eta(t) \Big) \; (**)$$

$$+ \Big(\underbrace{\frac{\partial^2 f}{\partial \dot{x}^2}(t, x_*(t), \dot{x}_*(t))}_{R(t)} \dot{\eta}(t), \dot{\eta}(t) \Big)$$

$$(*) + (**) = \frac{1}{2} \Big\{ \frac{\partial^2 f}{\partial \dot{x} \partial x}(t, x_*(t), \dot{x}(t)) + \Big[\frac{\partial^2 f}{\partial x \partial \dot{x}}(t, x_*(t), \dot{x}_*(t)) \Big]^T \Big\} = Q(t)$$

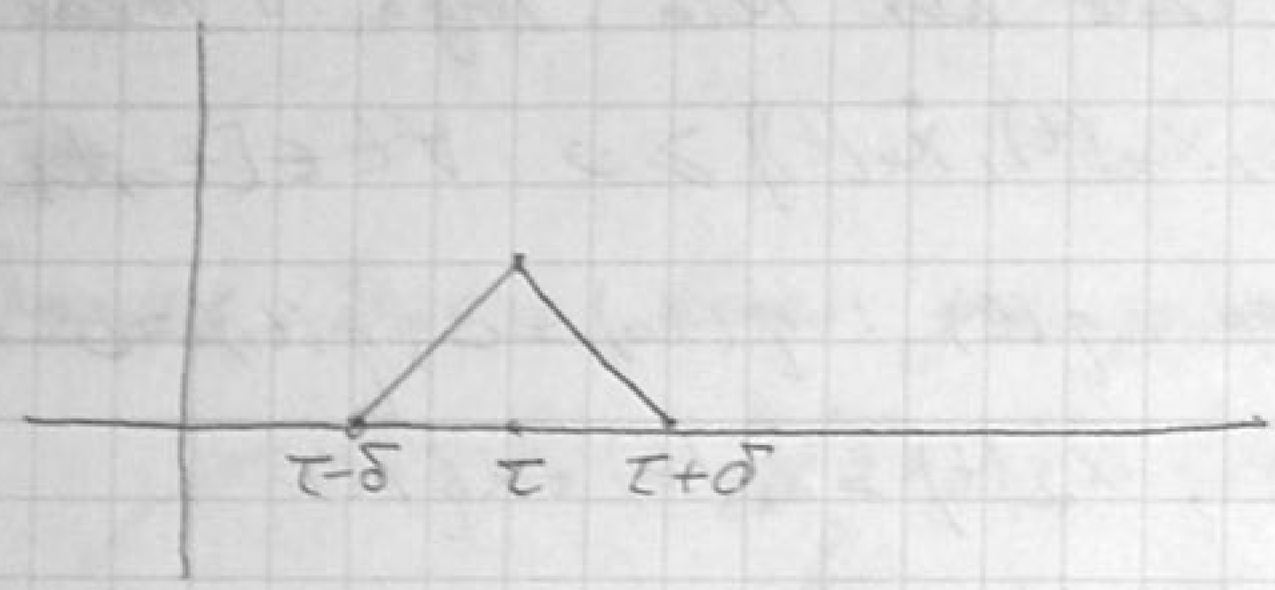
$$\Rightarrow \begin{cases} \ddot{I}(0) = \int_{t_0}^{t_1} \{ (P(t)\eta(t), \eta(t)) + 2(Q(t)\eta(t), \dot{\eta}(t)) + (R(t)\dot{\eta}(t), \dot{\eta}(t)) \} dt \geq 0 \\ \eta(t_0) = 0, \; \eta(t_1) = 0 \end{cases}$$

2-я вариация ф-ала

$\eta(t)$ – возмущение

$\forall \xi \in R^n$, $\tau \in (t_0, t_1)$ — точка непр. $\dot{x}_*(t)$; $\delta > 0$ (мала) : $t_0 < \tau - \delta$, $\tau + \delta < t_1$

$$\eta(t) = \begin{cases} 0, & t_0 \le t \le \tau - \delta,\ \tau + \delta \le t \le t_1 \\ \frac{1}{\sqrt{\delta}}\left(t - [\tau - \delta]\right)\xi, & \tau - \delta \le t \le \tau \\ \frac{1}{\sqrt{\delta}}\left([\tau + \delta] - t\right)\xi, & \tau \le t \le \tau + \delta \end{cases}$$



$$\dot{\eta}(t) = \begin{cases} 0, & t_0 \le t \le \tau - \delta \text{ и } \tau + \delta \le t \le t_1 \\ \frac{1}{\sqrt{\delta}}\xi, & \tau - \delta < t \le \tau \\ -\frac{1}{\sqrt{\delta}}\xi, & \tau < t \le (\tau + \delta) \end{cases}$$

$$\ddot{I}(0) = \int_{\tau-\delta}^{\tau+\delta} \left\{ \left(\frac{\delta - |t-\tau|}{\sqrt{\delta}}\right)^2 (P(t)\xi, \xi) + 2\frac{1}{\sqrt{\delta}} \cdot \frac{\delta - |t-\tau|}{\sqrt{\delta}} \cdot \right.$$

$$\left. \cdot (Q(t)\xi, \xi) + \frac{1}{\delta}(R(t)\xi, \xi) \right\} dt \ge 0$$

$$\frac{\delta - |t-\tau|}{\sqrt{\delta}} \le \sqrt{\delta}$$

$$\left| \int_{\tau-\delta}^{\tau+\delta} (P(t)\xi, \xi) \left(\frac{\delta - |t-\tau|}{\sqrt{\delta}}\right)^2 dt \right| \le \delta \int_{\tau-\delta}^{\tau+\delta} \left| (P(t)\xi, \xi) \right| dt = O(\delta)$$

$$\left| \int_{\tau-\delta}^{\tau+\delta} (Q(t)\xi, \xi) \left(\frac{\delta - |t-\tau|}{\sqrt{\delta}} \cdot \frac{1}{\sqrt{\delta}}\right) dt \right| \le \int_{\tau-\delta}^{t+\delta} \left| (Q(t)\xi, \xi) \right| dt = O(\delta)$$

$$\lim_{\delta \to +0} \ddot{I}(0) = (R(\tau)\xi, \xi) \ge 0 \Rightarrow R(\tau) \ge 0$$

<u>Теорема</u> $\exists\ x_*(t)$ — экстремаль в прост. задаче ВИ

1) $\frac{\partial^2 f}{\partial \dot{x}^2}(t, x, \dot{x}) > 0$

2) $\frac{\partial^2 f}{\partial x^2} - \frac{1}{4}\left\{ \frac{\partial^2 f}{\partial \dot{x} \partial x}(t, x, \dot{x}) + \left[\frac{\partial^2 f}{\partial x \partial \dot{x}}(t, x, \dot{x})\right]^T \right\} \times$

$$\times \left[\frac{\partial^2 f}{\partial \dot{x}^2}(t,x,\dot{x}) \right]^{-1} \cdot \left\{ \frac{\partial^2 f}{\partial \dot{x} \partial x}(t,x,\dot{x}) + \left[\frac{\partial^2 f}{\partial x \partial \dot{x}}(t,x,\dot{x}) \right]^T \right\} \geq 0$$

выпол. в окр. $\Gamma_{x_* \dot{x}_*} = \{ (t,x,\dot{x}) \in R^{2n+1} \ \| x - x_*(t) \| < \varepsilon, \ \| \dot{x} - \dot{x}_*(t) \| < \varepsilon, \ t \in [t_0, t_1] \}$

Тогда $x_*(t)$ доставляет слабый лок. min

<u>Док-во</u>: $\forall x(t),\ t \in [t_0, t_1],\ \eta(t) = x(t) - x_*(t)$

$$x_\alpha(t) = x_*(t) + \alpha \eta(t) \Rightarrow I(\alpha) = J(x_\alpha) = \int_{t_0}^{t_1} f(t, x_\alpha(t), \dot{x}_\alpha(t))\, dt$$

$I(0) = J(x_*)$,

$I(1) = J(x)$

$I'(0) = 0$

$I(1) = I(0) + \overset{0}{I'(0)} + \frac{1}{2} I''(\beta),\ \beta \in [0,1]$

$I(1) - I(0) = \frac{1}{2} I''(\beta)$

$P_\beta(t) = \frac{\partial^2 f}{\partial x^2}(t, x_\beta(t), \dot{x}_\beta(t))$

$R_\beta(t) = \frac{\partial^2 f}{\partial \dot{x}^2}(t, x_\beta(t), \dot{x}_\beta(t))$

$Q_\beta(t) = \frac{1}{2} \left\{ \frac{\partial^2 f}{\partial \dot{x} \partial x}(t, x_\beta(t), \dot{x}_\beta(t)) + \left[\frac{\partial^2 f}{\partial x \partial \dot{x}}(t, x_\beta, \dot{x}_\beta(t)) \right]^T \right\}$

$$\begin{cases} I(1) - I(0) = \frac{1}{2} \int_{t_0}^{t_1} \{ (P_\beta(t)\eta(t), \eta(t)) + 2(Q_\beta(t)\eta(t), \dot{\eta}(t)) + (R_\beta(t)\dot{\eta}(t), \dot{\eta}(t)) \}\, dt \\ \eta(t_0) = 0,\ \eta(t_1) = 0 \end{cases}$$

Рассм. $$\begin{cases} \tilde{J}(y) = \int_{t_0}^{t_1} \{ (P_\beta(t) y(t), y(t)) + 2(Q_\beta(t) y(t), \dot{y}(t)) + (R_\beta(t)\dot{y}(t), \dot{y}(t)) \}\, dt \to \min \\ y(t_0) = 0,\ y(t_1) = 0 \end{cases}$$

Покажем, что $y_*(t) = 0,\ t \in [t_0, t_1]$ — единств. реш. $\forall y(t) \neq 0$

~~[illegible]~~ $\Rightarrow \tilde{J}(y) - \tilde{J}(y_*) = \tilde{J}(y) =$

$$= \int_{t_0}^{t_1} \{ (R_\beta(t)\dot y(t), \dot y(t)) + 2(R_\beta(t)\dot y(t), R_\beta^{-1}(t)Q_\beta(t)y(t)) +$$

$$+ (\overbrace{R_\beta(t)R_\beta^{-1}(t)}^{E} Q_\beta(t)y(t), R_\beta^{-1}(t)Q_\beta(t)y(t)) +$$

$$+ (P_\beta(t)y(t)y(t)) - (Q_\beta(t)y(t), R_\beta^{-1}(t)Q_\beta(t)y(t)) \} dt =$$

$$= \int_{t_0}^{t_1} \{ \underbrace{(R_\beta(t)[\overbrace{\dot y(t) + R_\beta^{-1}(t)Q_\beta(t)y(t)}^{\neq 0}], \dot y(t) + R_\beta^{-1}(t)Q_\beta(t)y(t))}_{\geq 0} +$$

$$+ (\underbrace{[P_\beta(t) - Q_\beta(t)R_\beta^{-1}(t)Q_\beta(t)]}_{\geq 0} y(t), y(t)) \} dt > 0$$

$$\begin{cases} \dot y + R_\beta^{-1}(t)Q_\beta(t)y(t) = 0 \\ y(t_0) = 0, \ y(t_1) = 0 \end{cases} \Bigg| \Rightarrow y(t) = 0$$

$\Rightarrow I(1) - I(0) > 0$

$\Rightarrow J(x) - J(x_*) > 0$

<u>Пример</u> $J(x) = \int_{t_0}^{t_1} \{ (\underset{P^T \geq 0}{P} x(t), x(t)) + (\underset{R^T > 0}{R} \dot x(t), \dot x(t)) \} dt$

$$\frac{\partial^2 f}{\partial \dot x^2} = R > 0$$

$$\frac{\partial^2 f}{\partial x^2} = P > 0$$

<u>Теорема</u> $\exists\, x_*(t)$, $t \in [t_0, t_1]$ экстремаль, погруженная в обл G в центр. поле экстремалей, порожден. диффер. ф-цией действ. $S(t,x)$ $\exists\ \forall (t,x) \in G$ и $\forall w \in R^n$ $\frac{\partial^2 f}{\partial \dot x^2}(t, x, \underset{\uparrow\ v}{w}) \geq 0$

Тогда $x_*(t)$ доставляет сильный лок min

<u>Док</u>

$\triangleright$ $E(t, x, u(t,x), v) = f(t, x, v) - f(t, x, u(t,x)) -$

$$- \left(\frac{\partial f}{\partial \dot x}(t, x, u(t,x)), v - u(t,x) \right)$$

Разложим $f(t,x,v)$ в ряд Тейлора в окр-ти $u(t,x)$

$$f(t,x,v) = f(t,x,u(t,x)) + \left(\frac{\partial f}{\partial \dot{x}}(t,x,u(t,x)), v-u(t,x)\right) +$$
$$+\frac{1}{2}\left(\frac{\partial^2 f}{\partial \dot{x}^2}(t,x,w)(v-u(t,x)), (v-u(t,x))\right)$$

Сравним с $E(t,x,u(t,x),v)$

$$E(t,x,u(t,x),v) = f(t,x,v) - f(t,x,u(t,x)) -$$
$$-\left(\frac{\partial f}{\partial \dot{x}}(t,x,u(t,x)), v-u(t,x)\right)$$
$$\frac{1}{2}\left(\frac{\partial^2 f}{\partial \dot{x}^2}(t,x,w)(v-u(t,x)), (v-u(t,x))\right)$$

■

Следствие $\exists\ x_*(t),\ t\in[t_0,t_1]$ экстремаль, явл. элементом поле экстремалей, порожденного непр-дифф решение уравнение Гамильтона – Якоби $S(t,x)$. $\exists\ \forall (t,x)\in G$ и $\forall w\in R^n \Rightarrow$

$\Rightarrow \frac{\partial^2 f}{\partial \dot{x}^2}(t,x,w) \geq 0$

Тогда $x_*(t)$ доставляет сильный лок. min

§ Сопряж. точки. Условие Якоби

$$J(x) = \int_{t_0}^{t_1} f(t,x(t),\dot{x}(t))\,dt \to \min$$

$x(t_0)=x_0,\ x(t_1)=x_1$

$f(t,x,\dot{x})\in C^3,\ \exists x_*(t)$ – экстремаль

$\frac{\partial^2 f}{\partial \dot{x}^2}(t,x_*(t),\dot{x}_*(t)) > 0,\ t\in[t_0,t_1]$

Рассм.

$$\begin{cases} -\frac{d}{dt}(R(t)\dot{y}(t)) + (P(t)-\dot{Q}(t))y(t)=0 \\ y(t)=0,\ \dot{y}(t_0)=\xi \end{cases} \Longleftrightarrow$$

$$\Leftrightarrow \begin{cases} -R(t)\ddot{y}(t) - \dot{R}(t)\dot{y}(t) + (P(t) - \dot{Q}(t))y(t) = 0 \\ y(t_0) = 0, \dot{y}(t_0) = \xi \end{cases}$$

Если $\xi = 0 \Rightarrow y(t) = 0$ $\varnothing$ $\Rightarrow$ будем рассм. $\xi \neq 0$

<u>Опр</u> Если для некоторого $\xi \in \mathbb{R}^n$, $\xi \neq 0$ и $\tau \in (t_0, t_1]$:

$y(\tau) = 0$, где $y(t)$ — решение зад. Коши, то τ — точка сопряженной с t_0

<u>Теорема</u> Чтобы $x_*(t)$, $t \in [t_0, t_1]$ доставляла слаб. лок min, необходимо, чтобы на интервале (t_0, t_1) не $\exists$ точек сопряж. с t_0

Условие Якоби

<u>20.11.08</u>

<u>Док-во</u>: (от противного)

$$\begin{cases} \tilde{J}(y) = \frac{1}{2}\int\limits_{t_0}^{t_1} \{(P(t)y(t), y(t)) + 2(Q(t)y(t), \dot{y}(t)) + (R(t)\dot{y}(t), \dot{y}(t))\} dt \\ y(t_0) = 0, y(t_1) = 0 \end{cases}$$

$x_*(t) \Rightarrow$ необх. лок. min $\tilde{J}(y) \geq 0$ $\forall$ кусочно-дифф $y(t) \neq 0$

$\tilde{J}(y) \geq 0$ $\forall$ кусочно-дифф $y(t) \neq 0$ $\Rightarrow y_*(t) = 0$ доставляет абс. min

$$\tilde{J}(y) \geq \underset{\underset{0}{\parallel}}{J(y_*)}$$

От противного $\exists \tau \in (t_0, t_1)$ и $\xi \neq 0 : y(\tau) = 0$

решение ур-ния Якоби

$\Rightarrow \dot{y}(\tau) = 0$

Рассм. $\tilde{y}(t) = \begin{cases} y(t) & t_0 \leq t \leq \tau \\ 0 & \tau \leq t \leq t_1 \end{cases}$

$$\Rightarrow \tilde{J}(\tilde{y}) = \frac{1}{2}\int\limits_{t_0}^{\tau} \{(P(t)y(t), y(t)) + 2(Q(t)y(t), \dot{y}(t)) + (R(t)\dot{y}(t), \dot{y}(t))\} dt =$$

$$2\int_{t_0}^{\tau}(Q(t)y(t),\dot y(t))\,dt = \cancel{\int_{t_0}^{\tau}(Q(t)y(t),\dot y(t))\,dt} + \underbrace{\cancel{(Q(t)y(t),y(t))\Big|_{t_0}^{\tau}}}_{0} + \int_{t_0}^{\tau}\Big(\underbrace{\frac{d}{dt}(Q(t)y(t))}_{\dot Q(t)y(t)+\cancel{Q(t)\dot y(t)}},y(t)\Big)\,dt =$$

$$= -\int_{t_0}^{\tau}(\dot Q(t)y(t),y(t))\,dt$$

$$\int_{t_0}^{\tau}(R(t)\dot y(t),\dot y(t))\,dt = (R(t)\dot y(t),y(t))\Big|_{t_0}^{\tau} - \int_{t_0}^{\tau}\Big(\frac{d}{dt}[R(t)\dot y(t)],y(t)\Big)dt =$$

$$= -\int_{t_0}^{\tau}\Big(\frac{d}{dt}(R(t)\dot y(t)),y(t)\Big)dt$$

$$\Rightarrow \ominus\ \frac{1}{2}\int_{t_0}^{\tau}\underbrace{\Big(-\frac{d}{dt}(R(t)\dot y(t))+(P(t)-\dot Q(t))y(t),y(t)\Big)}_{=0}dt = 0$$

$$\Rightarrow \tilde J(\tilde y) = 0$$

Рассм. $g(t,y,\dot y) = \frac{1}{2}\{(P(t)y,y)+2(Q(t)y,\dot y)+(R(t)\dot y,\dot y)\}$

$$\frac{\partial^2 g}{\partial \dot y^2}(t,y,\dot y) = R(t) > 0, \quad t\in[t_0,t_1]$$

$$p(t) = \frac{\partial g}{\partial \dot y}(t,\tilde y(t),\dot{\tilde y}(t)) = Q(t)\tilde y(t)+R(t)\dot{\tilde y}(t)$$

Если $t \geq \tau$, $\tilde y(t)=0 \Rightarrow \tilde p(\tau+0)=0$

$\Rightarrow \tilde p(t)$ – непр. ф-ция

$\tilde p(t), \tilde y(t)$ – непр. ф-ции $\Rightarrow \tilde p(\tau-0)=0, \tilde y(\tau-0)=0$

$$\left.\begin{aligned} R(\tau)\dot{\tilde y}(\tau-0)=0 \Rightarrow \dot{\tilde y}(\tau-0)&=0\\ \dot{\tilde y}(\tau+0)&=0\end{aligned}\right] \Rightarrow \dot{\tilde y}(\tau)=0$$

противоречие

Условие Якоби – условие выполнение слаб. лок min

$$\begin{cases} -R(t)\ddot{y}(t) - \dot{R}(t)\cdot\dot{y}(t) + (P(t)-\dot{Q}(t))y(t) = 0 \\ \text{ур-ние Якоби} \\ y(t_0)=0,\ \dot{y}(t_0) = \xi \neq 0 \end{cases}$$

] $\dot{y}(t) = z(t)$

$$\begin{cases} \dot{y}(t) = z(t) \\ \dot{z}(t) = -R^{-1}(t)\dot{R}(t)z(t) + R^{-1}(t)(P(t)-\dot{Q}(t))y(t) \\ y(t_0)=0,\ z(t_0)=\xi\neq 0 \end{cases}$$

в виде лин. системы

$$\begin{pmatrix} 0 & E \\ R^{-1}(P-\dot{Q}) & -R^{-1}\dot{R} \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix}$$

$\Pi(t) \to 2n\times 2n$

$\to \Phi(t)$

$$\begin{cases} -\frac{d}{dt}[R(t)\dot{\Phi}(t)] + [P(t)-\dot{Q}(t)]\Phi(t) = 0 \\ \Phi(t_0) = 0 \\ \dot{\Phi}(t_0) = E \end{cases}$$

Лемма 1 $\tau\in(t_0,t_1]$ сопряжена с $t_0 \Leftrightarrow \Phi(\tau)$ – вырождена

Док-во:

Необх] $\tau\in(t_0,t_1]$ – сопряж с $t_0 \Rightarrow \exists\xi\in\mathbb{R}^n,\ \xi\neq 0$

$\Rightarrow y(t)$ – реш ур-ния Якоби и $y(\tau)=0$

$\boxed{y(t) = \Phi(t)\xi}$

$\Rightarrow \Phi(\tau)\xi = 0 \Rightarrow \Phi(\tau)$ – вырожденная м-ца

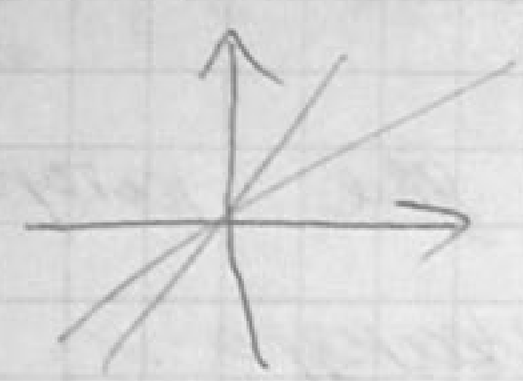
Дост] $\tau\in(t_0,t_1]$, $\Phi(\tau)$ – вырожд $\Rightarrow \Phi(\tau)\xi = 0 \Rightarrow$

$\Rightarrow \xi \neq 0$

$\Rightarrow y(t) = \Phi(t)\xi \Rightarrow y(\tau) = 0 \Rightarrow \tau$ – сопряж. точка

Лемма 2

$\exists$ есть $\dot{x}(t) = x(t)$, $x = Ce^t$



Рассм. $\dot{x}(t) = x(t) + q(x(t))$, $x \in \mathbb{R}^n$

$$q(\cdot) \in C^3(V),\ \exists K > 0 \quad \|q(x)\| \le K\|x\|^2,\ \forall x \in V$$

окр. нач. коорд.

Тогда $\forall l \in \mathbb{R}^n$ $\|l\| = 1$ $\exists!$ траектория $x(t)$:

$x(t) \to 0$, $\dfrac{\dot{x}(t)}{\|\dot{x}(t)\|} \to l$ при $t \to -\infty$

Док-во: Введем сферич. координаты $x = r\theta$, $\theta \in S_{n-1}$

□

$\dot{r}(t)\theta(t) + r(t)\dot{\theta}(t) = r(t)\theta(t) + q(r(t)\theta(t))$ $/\times\theta$

$\|q(x)\| \le K\|x\|^2$

$q(r\theta) = r^2\varphi(r,\theta)$

$\Uparrow$

$C^2([0,+\infty) \times S_{n-1})$

$(\theta(t), \theta(t)) = 1$

$(\theta(t), \dot{\theta}(t)) = 0$

$$\begin{cases} \dot{r}(t) = r(t) + r^2(t)\cdot(\theta(t), \varphi(r(t)\theta(t)) \ (=) \\ \dot{\theta}(t) = -r(t)(\theta(t), \varphi(r(t),\theta(t)))\theta(t) + r(t)\varphi(r(t),\theta(t)) \end{cases}$$

З. Коши

$(=)$ $r(1 + r(\theta,\varphi)) \Rightarrow \dot{r}(t) > 0$

$\exists \tilde{V} \subset V$ где $r(t) \downarrow 0$ $t \to -\infty$ $\Rightarrow$ в $\tilde{V}$

$$\frac{d\theta}{dr}(r) = \underbrace{\frac{-(\theta(r), \varphi(r,\theta(r)) + \varphi(r,\theta(r))}{1 + r\cdot(\theta(r), \varphi(r,\theta(r))}}_{\Uparrow\ C(\tilde{V})} = \psi(r,\theta)$$

$x(t) \to 0$

$\dfrac{\dot{x}(t)}{\|\dot{x}(t)\|} \to l$ $\quad t \to -\infty$ $\quad \Leftrightarrow \theta(0) = l$

$$\begin{cases} \dot\Theta(\tau) = \varphi(\tau, \Theta(\tau)) \\ \Theta(0) = e \end{cases}$$

$\boxtimes$

<u>Теорема</u> $\exists x_*(t)$ – экстремаль в ПЗВИ для которой выполнено усиленное условие Лежандра.
Пусть $(t_0, t_1]$ не содержит точек, сопряж с точкой t_0, тогда $x_*(t)$ доставляет слабый лок min

p.s.
(t_0, t_1) – усл. Якоби
$(t_0, t_1]$ – усиленное условие Якоби

<u>Док-во:</u> $V = \{(t, x, \dot x) \in \mathbb{R}^{2n+1} ; \|x - x_*(t)\| < \delta$,
$\|\dot x - \dot x_*(t)\| < \delta, t \in [t_0, t_1]\}$

$\overline{V}$ – замыкание

$\exists$ производные $< M$

$\forall x(t) ; \eta(t) = x(t) - x_*(t), \eta(t_0) = 0, \eta(t_1) = 0.$

$$x_\alpha(t) = x_*(t) + \alpha\eta(t)$$

$$I(\alpha) = I(x_\alpha) = \int_{t_0}^{t_1} f(t, x_\alpha(t), \dot x_\alpha(t))\, dt$$

$$I(0) = J(x_*), I(1) = J(x), \dot I(0) = 0$$

$$J(x) - J(x_*) = I(1) - I(0) = \cancel{\dot I(0)} + \frac{1}{2}\ddot I(\beta), \quad \beta \in [0, 1]$$

$$\ddot I(\beta) = \int_{t_0}^{t_1} \Big\{ \Big(\frac{\partial^2 f}{\partial x^2}(t, x_\beta(t), \dot x_\beta(t))\eta(t), \eta(t)\Big) +$$

$$+ \Big(\frac{\partial^2 f}{\partial \dot x \partial x}(t, x_\beta(t), \dot x_\beta(t))\dot\eta(t), \eta(t)\Big) +$$

$$+ \Big(\frac{\partial^2 f}{\partial x \partial \dot x}(t, x_\beta(t), \dot x_\beta(t))\eta(t), \dot\eta(t)\Big) +$$

$$+ \Big(\frac{\partial^2 f}{\partial \dot x^2}(t, x_\beta(t), \dot x_\beta(t))\dot\eta(t), \dot\eta(t)\Big)\Big\} dt =$$

$$= \int_{t_0}^{t_1} \{ (P(t)\eta(t), \eta(t)) + 2(Q(t)\eta(t), \dot\eta(t)) + (R(t)\dot\eta(t), \dot\eta(t)) \} dt + 2\Delta$$

$$\Delta = \frac{1}{2}\int_{t_0}^{t_1} \{ ([\frac{\partial^2 f}{\partial x^2}(t, x_\beta(t), \dot x_\beta(t)) - \frac{\partial^2 f}{\partial x^2}(t, x_*(t), \dot x_*(t))] \eta(t), \eta(t)) + \ldots$$

...

$$|\Delta| \le \tilde M \cdot \max \{ \max_{[t_0,t_1]} \|\eta(t)\|, \max_{[t_0,t_1]} \|\dot\eta(t)\| \} \cdot \int_{t_0}^{t_1} (\dot\eta(t), \dot\eta(t)) dt$$

$$\tilde M = \frac{1}{2} n\sqrt{n} M \{ (\frac{t_1 - t_0}{\sqrt 2} + \frac{3}{2})^2 + \frac{7}{4} \}$$

За счет Δ величина имеет больший порядок малости

27.11.08

$$|\Delta| \le \tilde M \cdot \max \{ \max_{[t_0,t_1]} \|\eta(t)\|, \max \|\dot\eta(t)\| \} \int_{t_0} (\dot\eta(t), \dot\eta(t)) dt$$

$$\int_{t_0}^{t_1} \{ (P(t)\eta(t), \eta(t)) + 2(Q(t)\eta(t), \dot\eta(t)) + (R(t)\dot\eta(t), \dot\eta(t)) \} dt + 2\Delta$$

$$2\int_{t_0}^{t_1} (Q(t)\eta(t), \dot\eta(t)) dt = \int_{t_0} (Q(t)\eta(t), \dot\eta(t)) dt + (Q(t)\eta(t), \eta(t)) \Big|_{t_0}^{t_1} -$$

$$- \int_{t_0}^{t_1} (\frac{d}{dt}[Q(t)\eta(t)], \eta(t)) dt =$$

$$\frac{d}{dt}[Q(t)\eta(t)] = \dot Q(t)\eta(t) + Q(t)\dot\eta(t)$$

$$= -\int_{t_0}^{t_1} (\dot Q(t)\eta(t), \eta(t)) dt$$

$$J(x) - J(x_*) = \frac{1}{2}\int_{t_0}^{t_1} \{ ([P(t) - \dot Q(t)]\eta(t), \eta(t)) + (R(t)\dot\eta(t), \dot\eta(t)) \} dt + \Delta$$

$$\tilde J(y) = \int_{t_0}^{t_1} \{ ([P(t) - \dot Q(t)] y(t), y(t)) + ([R(t) - \varepsilon^2 E] \dot y(t), \dot y(t)) \} dt$$

$y(t_0) = 0$, $y(t_1) = 0$

$R(t) > 0$, $t \in [t_0, t_1]$

Б

$[R(t)-\tau^2 E]>0$

$(t_0, t_1]$ $\Phi(t)$ невырождена по лемме 1

$$\begin{cases} -\frac{d}{dt}\left[\left(R(t)-\tau^2 E\right)\dot{\Phi}(t)\right]+\left(P(t)-\dot{Q}(t)\right)\Phi(t)=0 \\ \Phi(t_0)=0 \\ \dot{\Phi}(t_0)=E \end{cases}$$

$$\Rightarrow \begin{cases} \tilde{J}(y)=\int_{t_0}^{t_1}\left\{(P_0(t)y(t),y(t))+(R_\tau(t)\dot{y}(t),\dot{y}(t))\right\}dt \\ y(t_0)=0,\ y(t_1)=0 \end{cases}$$

$$-\frac{d}{dt}\left[R_\tau(t)\dot{\Phi}(t)\right]+P_0(t)\Phi(t)=0$$

$\Phi(t_0)=0$, $\dot{\Phi}(t_0)=E$

$\Phi_\tau(t)$ невырож, $t\in[t_0,t_1]$

Введем

$$W(t)=-R_\tau(t)\dot{\Phi}_\tau(t)\Phi_\tau^{-1}(t),\ t\in[t_0,t_1]$$

① $\dot{W}(t)-W(t)R_\tau^{-1}(t)W(t)+P_0(t)=0$

$$\dot{W}(t)=-\frac{d}{dt}\left[R_\tau(t)\dot{\Phi}_\tau(t)\right]\Phi_\tau^{-1}(t)-R_\tau(t)\dot{\Phi}_\tau(t)\frac{d}{dt}\left[\Phi_\tau^{-1}(t)\right]$$

$$\frac{d}{dt}\left[\Phi_\tau^{-1}(t)\right]=-\Phi_\tau^{-1}(t)\dot{\Phi}_\tau(t)\Phi_\tau^{-1}(t)$$

$$\Rightarrow \dot{W}(t)=-P_0(t)\Phi_\tau(t)\Phi_\tau^{-1}(t)+R_\tau(t)\dot{\Phi}_\tau(t)\Phi_\tau^{-1}(t)\underbrace{\dot{\Phi}_\tau(t)}_{R_\tau^{-1}(t)R_\tau(t)\,\vee}=$$

$$=-P_0(t)+WR_\tau^{-1}W$$

② $W(t)$ в окр t_0

$$\begin{cases} \Phi_\tau(t)=(t-t_0)\left(E+(t-t_0)\Omega_1(t)\right) \\ \dot{\Phi}_\tau(t)=E+(t-t_0)\Omega_2(t) \end{cases}$$

$$\hookrightarrow \Phi_\tau^{-1}(t)=\frac{1}{t-t_0}\left(E+(t-t_0)\Omega_3(t)\right)$$

$\Omega_1(t), \Omega_2(t), \Omega_3(t)\in C$ (в окр. t_0)

$$W(t) = -\frac{1}{t-t_0} R_2(t)\Omega(t)$$

$$\Rightarrow \Omega(t) = -R_2(t)\left(\Omega_2(t) + \Omega_3(t) + (t-t_0)\Omega_2(t)\Omega_3(t)\right)$$

③ Как по реш. ур-ние Риккати восстан. реш. ур-ние Якоби

<u>Лемма 3</u> Пусть $\tilde{W}(t)$ – решение ур-ние Риккати, удовл. $\tilde{W}(t) = \frac{1}{t-t_0} R_2(t) + \underset{\substack{\Uparrow \\ C([t_0,t_1])}}{\tilde{\Omega}(t)}$

$\Rightarrow \exists!\ Z(t)$ удовл. ур-нию $R_2(t)\dot{Z}(t) + \tilde{W}(t)Z(t) = 0$

$Z(t_0) = 0$, $\dot{Z}(t_0) = E$, которая удовл. ур-нию Якоби

<u>Док-во</u>: $\rceil\ v = t-t_0$, $\tau = \ln(t-t_0)$

$$\begin{cases} \frac{dz}{d\tau}(\tau) = -v(t)R_2^{-1}(\tau)\cdot\tilde{W}(\tau)Z(\tau) \\ \frac{dv}{d\tau} = v(\tau) \end{cases}$$

$$\begin{cases} \frac{dz}{d\tau}(\tau) = Z(\tau) - v(\tau)R_2^{-1}(\tau)\tilde{\Omega}(\tau)Z(\tau) \\ \frac{dv}{d\tau}(\tau) = v(\tau) \end{cases}$$

$Z(\tau) \to 0$, $v(\tau) \to 0$ при $\tau \to \infty$

$$\underbrace{-\frac{d}{dt}\left[R_2(t)\dot{Z}(t)\right] + P_0(t)Z(t)}_{\text{левая часть ур-ние Якоби}} = \frac{d}{dt}\left[\tilde{W}(t)Z(t)\right] + P_0(t)Z(t) =$$

$$= \underset{\substack{\parallel \\ (\dot{\tilde{W}}(t) + P_0(t))Z(t)}}{\dot{\tilde{W}}(t)Z(t) + P_0(t)Z(t)} + \tilde{W}(t)\dot{Z}(t) =$$

$$= \tilde{W}(t)R_2^{-1}(t)\tilde{W}(t)Z(t) - \tilde{W}(t)R_2^{-1}(t)\tilde{W}(t)Z(t) = 0$$

$W(t) = W^T(t)$, $t \in (t_0, t_1]$

$\dot{W}^T(t) - W^T(t)R_2^{-1}(t)W^T(t) + P_0(t) = 0$

$W^T(t)$ – решение ур-ние Риккати

$W^T(t) = -\frac{1}{t-t_0}R_2(t) + \Omega^T(t)$

$W^*(t)$ по Лемме 3

$\downarrow$

$\tilde{\Phi}_z(t)$: $R_z(t)\,\dot{\tilde{\Phi}}_z(t) + W^T(t)\,\tilde{\Phi}_z(t) = 0$, $\tilde{\Phi}_z(t_0) = 0$, $\dot{\tilde{\Phi}}_z(t_0) = E$

$\downarrow$

реш. ур-ния Якоби

$W(t) = -R_z(t)\,\dot{\Phi}_z(t)\,\Phi_z^{-1}(t)$

$0 = (W - \tilde{W})\Phi_z$

$\Rightarrow W = \tilde{W}$

$$\begin{cases} \tilde{J}(y) = \int_{t_0}^{t_1} \{(P_0(t)y(t), y(t)) + (R_z(t)\dot{y}(t), \dot{y}(t))\}\,dt \\ y(t_0) = 0,\ y(t_1) = 0 \end{cases}$$

$y(t) = (t - t_0)\rho(t)$

$\exists\, \rho(t) \in C^1[t_0, t_1]$

$W(t) = -R_z(t)\,\dot{\Phi}_z(t)\,\Phi_z^{-1}(t)$

$$0 = (W(t_1)y(t_1), y(t_1)) - \lim_{t \to t_0} (W(t)y(t), y(t)) = \int_{t_0}^{t_1} \frac{d}{dt}(W(t)y(t), y(t)) =$$

$$= \int_{t_0}^{t_1} (\dot{W}(t)y(t), y(t))\,dt + 2\int_{t_0}^{t_1} (W(t)y(t), \dot{y}(t))\,dt$$

$$\Rightarrow \tilde{J}(y) = \int_{t_0}^{t_1} \{ P_0(t)y(t), y(t)) + (R_z(t)\dot{y}(t), \dot{y}(t)) + (\dot{W}(t)y(t), y(t)) +$$

$$+ 2(W(t)y(t), \dot{y}(t))\}\,dt =$$

$$= \int_{t_0}^{t_1} \{(R_z(t)\dot{y}(t), \dot{y}(t)) + 2(W(t)y(t), \dot{y}(t)) + (W(t)R_z^{-1}(t)W(t)y(t), y(t))\}\,dt$$

$$\tilde{J}(y) = \int_{t_0}^{t_1} (R_z^{1/2}(t)\dot{y}(t) + R_z^{-1/2}(t)y(t),\ R_z^{1/2}(t)\dot{y}(t) + R_z^{-1/2}(t)W(t)y(t))\,dt \quad (*)$$

$\exists\, y_*(t)$ обращающ. $(*)$ в нуль

$\uparrow$

экстремаль $\Rightarrow$ удовл. ур-нию Эйлера

$-\frac{d}{dt}[R_z(t)\dot{y}(t)] + P_0(t)y(t) = 0$

$\Rightarrow \dot{y}(t) + R_z^{-1}(t)W(t)y(t) = 0$ для всех $t \in [t_0, t_1]$

$\exists\, t = t_0 \Rightarrow y(t_0) = 0 \Rightarrow \dot{y}(t_0) = 0$

$\Rightarrow -R_z(t)\ddot{y}(t) - \dot{R}_z(t)\dot{y}(t) + P_0(t)y(t) = 0$

$\Rightarrow \; y_*(t) = 0$

Если $\forall y(t) = 0 \Rightarrow \tilde{J}(y) > 0$

$\exists\, \varepsilon_*$

$$\int_{t_0}^{t_1} \{ ([P(t) - \dot{Q}(t)]\, y(t), y(t)) + (R(t)\dot{y}(t), \dot{y}(t)) \} dt > \varepsilon_*$$

$$> \varepsilon_*^2 \int_{t_0}^{t_1} (\dot{y}(t), \dot{y}(t))\, dt$$

$\Rightarrow J(x) - J(x_*) > 0$

Лемма 4 $\exists\, x_*(t),\ t \in [t_0, t_1]$ — экстремаль и выполн. усиленные условие Якоби и Лежандра

Тогда $x_*(t)$ можно окружить полем экстремалей

Д-во:
$$\frac{\partial^2 f}{\partial \dot{x}^2}(t, x(t), \dot{x}(t))\ddot{x}(t) + \frac{\partial^2 f}{\partial \dot{x}\partial x}(t, x(t), \dot{x}(t))\,\dot{x}(t) + \frac{\partial^2 f}{\partial \dot{x}\partial t}(t, x(t), \dot{x}(t)) - \frac{\partial f}{\partial x}(t, x(t), \dot{x}(t)) = 0$$

$$\Rightarrow \exists\, V \quad \underbrace{\{ t, x_*(t), \dot{x}_*(t),\ t \in [t_0, t_1] \}}_{\Gamma_{x_*}} \in R^{2n+1}$$

$$\frac{\partial^2 f}{\partial \dot{x}^2}(t, x, \dot{x}) > 0 \quad \forall (t, x, \dot{x}) \in V$$

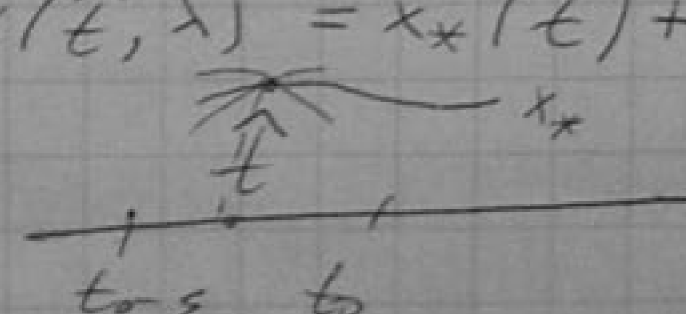
$$\begin{cases} \dot{x} = y \\ \dot{y} = \Lambda(t, x, y) \end{cases} \in C^1$$

$$\Lambda(t, x, y) = \left[\frac{\partial^2 f}{\partial \dot{x}^2}(t, x, y) \right]^{-1} \left\{ \frac{\partial f}{\partial x}(t, x, y) - \frac{\partial^2 f}{\partial x \partial t}(t, x, y) - \frac{\partial^2 f}{\partial \dot{x}\partial x}(t, x, y)\, y \right\}$$

По теореме $\exists$-ния и непрер. зав-ти реш. диф. ур- от нач. условий $\exists\, \varepsilon > 0$ и $\delta > 0$: ⓐ $x_*(t)$ продолжима на $[t_0 - \varepsilon, t_1 + \varepsilon]$

ⓑ $\forall \lambda \in R^n \quad \|\lambda\| < \delta$ на $[t_0 - \varepsilon, t_1 + \varepsilon]$

$\exists$ решение $x(t, \lambda)$: $\quad x(\hat{t}, \lambda) = x_*(\hat{t})$
$\dot{x}(\hat{t}, \lambda) = \dot{x}_*(\hat{t}) + \lambda, \ \hat{t} \in (t_0 - \varepsilon, t_0)$



$$\left.\frac{\partial x(t,\lambda)}{\partial\lambda}\right|_{\lambda=0} = D(t)$$

$$0 = \frac{\partial}{\partial\lambda}\left[-\frac{d}{dt}\frac{\partial f}{\partial\dot x}(t, x(t,\lambda), \dot x(t,\lambda)) + \frac{\partial f}{\partial x}(t, x(t,\lambda), \dot x(t,\lambda))\right]\Bigg|_{\lambda=0}$$

$$\Rightarrow -\frac{d}{dt}\left[R(t)\dot D(t) + Q(t)D(t)\right] + Q(t)\dot D(t) + P(t)D(t) = 0$$

$D(t)$ удовл. уравнению Якоби

$$D(\hat t) = \frac{\partial x}{\partial\lambda}(\hat t,\lambda)\Big|_{\lambda=0}, \quad \dot D(\hat t) = \frac{\partial\dot x}{\partial\lambda}(\hat t,\lambda)\Big|_{\lambda=0} = E$$

По условию выполнено усиленное условие Якоби

$\Rightarrow \Phi(t)$ невырожд. на $(t_0, t_1]$

По теор. о непр. зависимост

$\hat t \to t_0$ $D(t)$ невырож. для $[t_0, t_1]$

Рассм. отобр.

$\psi(t,\lambda) = (t, x(t,\lambda))$ в точке $(s,0)$, $s\in[t_0,t_1]$

$\psi(s,0) = (s, x_*(s))$

$$\det\dot\psi(s,0) = \det\begin{vmatrix} 1 & 0 \\ \frac{\partial x}{\partial t}(s,0) & \frac{\partial x}{\partial\lambda}(s,0)\end{vmatrix} = \det D(s) \neq 0$$

По теореме об обратной ф-ии $\exists\ \delta = \delta(s) > 0$

$|\tau - s| < \delta$ и $\|z - x_*(s)\| < \delta$, то $\exists!\ \lambda = \lambda(\tau,z)$:

$\psi(\tau, \lambda(\tau,z)) = (\tau, z)$

$\Updownarrow$

$x(\tau, \lambda(\tau,z)) = z,$

$(\tau,z) \in$ окр. $\{(t, x_*(t)), t\in[t_0,t_1]\}$

$\exists\ \delta_0 : \forall (\tau,z)$ $\|z - x_*(\tau)\| < \delta_0$

$\exists!\ \lambda = \lambda(\tau,z)$: $x(\tau, \lambda(\tau,z)) = z_1$

Теорема $\exists\, x_*(t)$ - экстремаль в ПЗВИ и выполнено усиленное условие Якоби

$\exists\ \forall w \in R^n \ \frac{\partial^2 f}{\partial \dot{x}^2}(t, x_*(t), w) > 0 \quad \forall t \in [t_0, t_1]$

Тогда $x_*(t)$ доставляет сильный лок min