Численные методы оптим. упр-ния,

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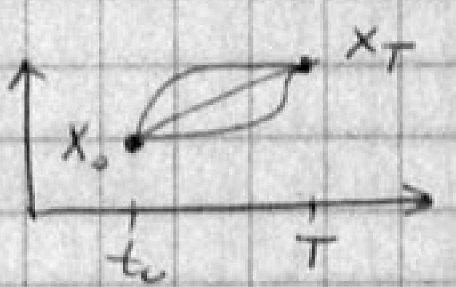
3 курс 6 семестр

19.02.09 ~~$X(t) = e^{(t-t_0)A}$~~ [illegible]

§1 Методы решения краевых задач. Принцип Max

$$(1)\ \begin{cases} \dot{x} = f(t,x,u), & t \in [t_0, T] \\ x(t_0) = x_0 \\ x(T) = x_T \end{cases}$$



$u(t) \in V$

$x \in R^n,\ u \in R^m$ V – замкн. мн-во

$$J(u) = \int_{t_0}^{T} f_0(t,x,u)\,dt \qquad (2)$$

Ф-ие Гамильтона – Понтрягина

$$H(\psi, x, u, t) = -f_0(t,x,u) + \langle \psi, f(t,x,u) \rangle \qquad (3)$$

$$\dot{\psi}_i = \frac{\partial f_0}{\partial x_i} - \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_i} \psi_j \qquad \text{Сопрях. система} \qquad (4)$$

$$\Rightarrow H(\psi(t), x(t), u(t), t) = \max_{u \in V} H(\psi(t), x(t), u, t) \qquad (5)$$

← принцип max

$\exists\ u(t) = V(x(t), \psi(t), t)$

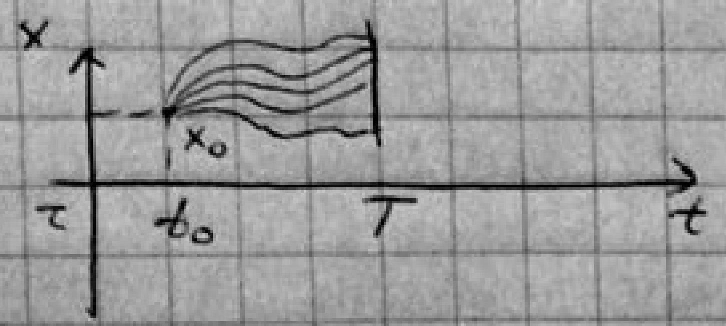
Подставим в (1), (4)

$$(7)\ \begin{cases} \dot{x} = f(t, x, V(x(t), \psi(t), t)) \\ \dot{\psi} = f_1(x, V(x(t), \psi(t), t), t, \psi) \end{cases}$$

краевая задача принципа Max

$$(5)\ \begin{cases} x(t_0) = x_0 \\ x(T) = x_T \end{cases}$$

$$J(u) = \Phi(x(T))$$



Введем ф-ию Гамильтона – Понтрягина

$$H(\psi, x, u, t) = \langle \psi, f(t,x,u) \rangle$$

$$\begin{cases} \dot{\psi}_i(t) = -\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_i} \psi_j \\ \psi_i(T) = -\Phi'_j(x(T)) \end{cases}$$

Пример 1 Движение маятника.

$$\begin{cases} \overset{\circ}{x}_1 = x_2 \\ \dot{x}_2 = -\underbrace{\sin x_1} + u \end{cases}$$

Нелин. зад.

x_1 - отклонение маятника
x_2 - скорость

$x_1(0) = x_{10}$ $\quad |u(t)| \leq 1$
$x_2(0) = x_{20}$

$$J(u) = |x_1(T)|^2 + |x_2(T)|^2$$

$$H(\psi, x, u, t) = \psi_1 x_2 + \psi_2 \cdot (-\underbrace{\sin x_1} + u)$$

$$u(t) = \text{sign}\, \psi_2(t)$$

$$\begin{cases} \overset{\circ}{\psi}_1 = \psi_2 \underbrace{\cos x_1} \\ \overset{\circ}{\psi}_2 = -\psi_2 \end{cases}$$

$$\psi_1(T) = -2x_1(T)$$

$$\psi_2(T) = -2x_2(T)$$

$$\begin{cases} \overset{\circ}{x}_1 = x_2 \\ \overset{\circ}{x}_2 = -\underbrace{\sin} x_1 + \text{sign}\, \psi_2(t) \end{cases}$$

Краевая задача принципа max

$x_1(0) = x_{10}$
$x_2(0) = x_{20}$

$\sin \equiv x_1$

Пример 2

$$J(u) = \int_0^T u^2(t)\, dt$$

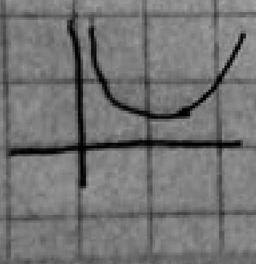
$\overset{\circ}{x}_1 = x_2$
$\dot{x}_2 = -\sin x_1 + u$

$x_1(0) = x_{10}$
$x_2(0) = x_{20}$

$[0, T]$

$$H(\psi, u, x, t) = -u^2 + \psi_1 x_2 + \psi_2(-\sin x_1 + u) = u^2 + u\psi - \psi_2 \sin x_1$$

$$u(t) = \frac{1}{2}\psi_2(t)$$



$$\begin{cases} \dot{\psi}_1 = \psi_2 \cos x_1 \\ \dot{\psi}_2 = -\psi_1 \end{cases}$$

ПМП:

$$\begin{cases} \dot{x} = f(x, \psi, v(x, \psi, t), t) \\ \dot{\psi} = f_1(x, \psi, t) \end{cases}$$

$x(t_0) = x_0$ ⓝ

$x(T) = x_T$

$\psi(t_0) = z$ ⓝ

$\tilde{x}(t)$, $\tilde{x}(T)$

Расм. $F = \tilde{x}(T) - x_T$ невязка

$$F = F(z)$$

$$F(z) = 0$$

$$F = \tilde{\psi}(T) - \psi(T) = \tilde{\psi}(T) + \Phi'(x(T))$$

$z^{(k)} = z^{(k-1)} + \alpha_k \Delta z^{(k-1)}$ $k = 1, 2, \dots$

$$\Delta z^{(k-1)} = -\left(F'(z^{(k-1)})\right)^{-1} F(z^{(k-1)})$$

F' $\frac{\partial F_i}{\partial z_j}$ М-ца 2-х производных

$$0 < \alpha_k < 1$$

$$S(z) = \sum_{i=1}^{n} F_i(z)$$

$$S(z^{(k)}) < S(z^{(k-1)}$$

Возьмем $\alpha_k = 1$ Смотрим, выполняется или нет

Если нет, то берем $\alpha_k = \frac{1}{2}$

в общем случае это метод Ньютона, но модифицируемый

$$S(z^{(k)}) < \varepsilon$$

$$\frac{\partial F}{\partial z_j} \simeq \frac{F_i(x_1, \dots, x_j + \delta_j, \dots x_n) - F_i(x_1, \dots, x_j, \dots, x_n)}{\delta_j}$$

5.03.09

§2 Метод последовательных приближений.

$$(1) \begin{cases} \dot{x} = f(t,x,u) \qquad [t_0, T] \\ x(t_0) = x_0 \\ u \in U \end{cases}$$

$$(2) \quad I(u) = F(x(T))$$

Ψ

~~$H(\psi)$~~ $H(\psi, x, u, t) = \langle \psi, f(t,x,u) \rangle$

$$(3) \begin{cases} \dot{\psi} = \dfrac{-\partial H(\psi, x, u, t)}{\partial x} \\ \psi(T) = -F'(x(T) \end{cases}$$

сопряж система

$$(4) \quad H(\psi, x, u(x, t, \psi), t) = \max_{u \in U} H(\psi, x, u, t)$$

$F(x) = 0$, x^0 – задано

$x^{(n+1)} = F(x^{(n)})$

$u^1(t)$ – задано

1) (1) $u = u^{(k)}(t)$ $[t_0, T]$ $\Rightarrow$ [illegible] $x^{(k)}(t)$

2) (3) $u^k(t), x^{(k)}(t)$ $\Rightarrow$ $\psi^k(t)$

3) (4) [illegible] $x^{(k)}(t), \psi^k(t)$ $\Rightarrow$ $u^{(k+1)}(t)$

определется неоднозначно

4) $|u^{(k+1)}(t) - u^{(k)}(t)| \le \varepsilon$

$u^{(k+1)}(t) = F(u^{(k)})$

$$(*) \begin{cases} \dot{x} = Ax + \ddot{u} \\ x(t_0) = x_0 \\ u \in U \\ I(u) = \langle c, x(T) \rangle \end{cases} \qquad \begin{cases} \dot{\psi} = -A^* \psi \\ \psi(T) = -c \end{cases}$$

$u^{(1)}(t) \to x^{(1)}(t), \psi^{(1)}(t)$

$$\langle \psi^{(1)}(t), u^{(2)}(t) \rangle = \max_{u \in U} \langle \psi^{(1)}(t), u \rangle$$

$\Rightarrow u^{(2)}(t), x^{(2)}(t)$

Способы улучшения сх-ти

① $\dot{x} = \varepsilon f(t, x, u) \qquad \varepsilon \to 0$

слабо управл. система

$\dot{x} = f(t, x, \varepsilon, u)$

$\varepsilon \to 0 \quad u^{(1)}(t) \to \dots \to u^{(N)}(t)$

$\varepsilon = 10^{-6}$ ↗

$\varepsilon = 10^{-5} \quad u^{(N)}(t) \to \dots \to u^{(M)}(t)$

⋮

$\varepsilon = 1 \quad u^{(N^*)}$

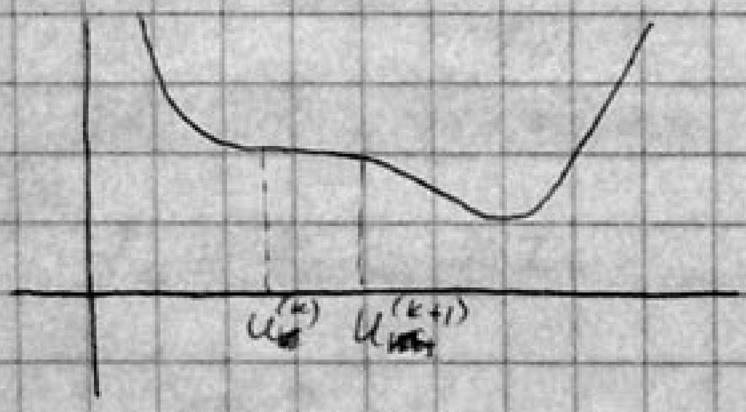
② $u^{(k+1)}(t) = (1-\alpha)\, u^{(k)}(t) + \alpha F^{(k)}(u^{(k)}(t))$

$0 \le \alpha \le 1$ Если $\alpha = 1$ то исходный метод

$\bar{U}$ - вып.

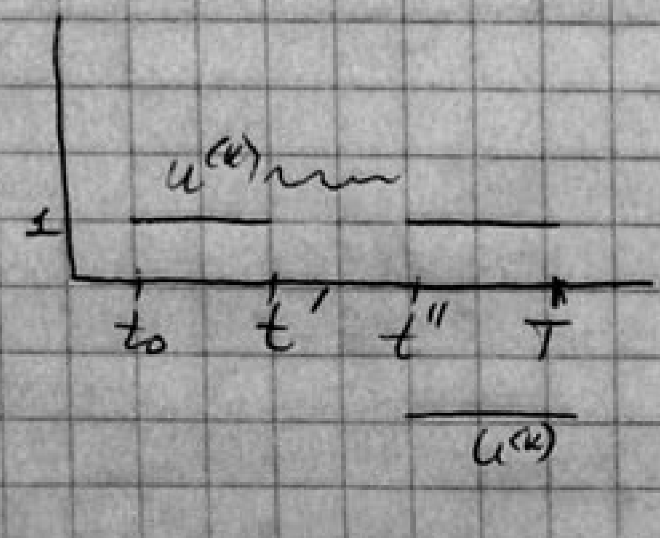
$I(u) = \Phi(\alpha)$

$|I(u^{(k+1)}) - I(u^{(k)})| \le \varepsilon$



③ $u^{(k+1)}(t) = F(u^{(k)}(t)),\ t \in [t', t']$

$u^{(k+1)}(t) = u^{(k)}(t),\ t \notin [t', t'']$



$t'' = T$

12.03.09

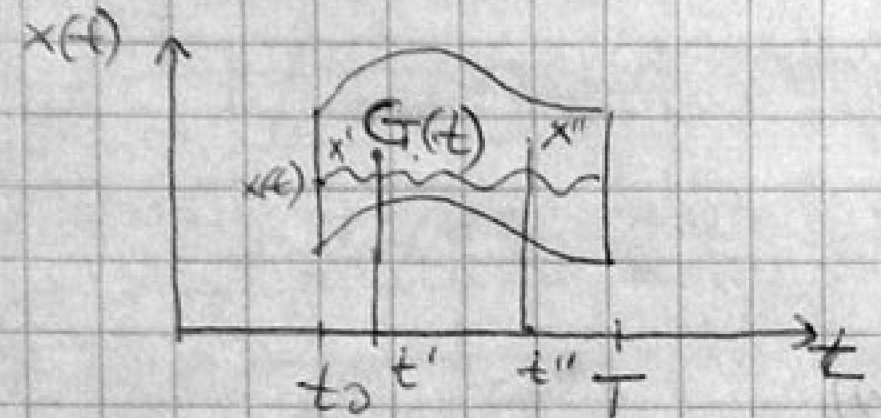
§3 Метод Локальных Вариаций

$\dot{x} = f(t,x,u)$ (1)

$[t_0, T]$

(2) $x(t) \in G(t)$, $u(t) \in U(t)$

многознач. отображ.



$x(t) \in G(t)$ 30% с фазовыми огранич.

$u(t) \in U(t)$

$$I(u) = \int_{t_0}^{T} f_0(t,x,u)\,dt \qquad (3)$$

$t_0 \le t' \le t'' \le T$

x' x''

$x' \in G(t')$

$x'' \in G(t'')$

$[t', t'']$ $x' \to x''$

$$\Delta I = \int_{t'}^{t''} f_0(t,x,u)\,dt$$

1) $\exists ?\ u(t) \in U(t) : x' \to x''$

2) $U(t) \approx U\left(\frac{t'+t''}{2}\right)$

$f(t,x,u) \approx f\left(\frac{t'+t''}{2}, \frac{x'+x''}{2}, u\right)$

a) $n - x$; $m - u$ $n = m$

$$x'' - x' = f\left(\frac{t'+t''}{2}, \frac{x'+x''}{2}, u\right)(t''-t')$$

$$\Delta I = f_0\left(\frac{t'+t''}{2}, \frac{x'+x''}{2}, u\right)(t''-t')$$

б) $m > n$

ⓝ ур-ний

$m-n$ компонент вектора u

делаем тоже самое

в) $m < n$

$h = 2m$

$h = t'' - t'$, $t^* = t' + \frac{h}{2}$

$[t', t^*]$, $[t^*, t'']$

$x^* = x(t^*)$

u_2

$x^* - x' = 0{,}5h\, f(t', x', u_1)$

$x'' - x^* = 0{,}5h\, f(t^*, x^*, u_2)$

$x'' - x' = 0{,}5h\left(f(t', x', u_1) + f\left(t' + \frac{h}{2}, x' + 0{,}5h\, f(t', x', u_1), u_2\right)\right)$

1) элементарные операции (можно ли прийти в др. точку, используя опред. упр-ние

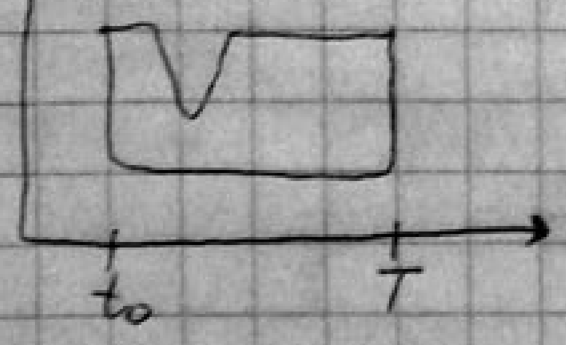
2) $x(t) \in G(t)$

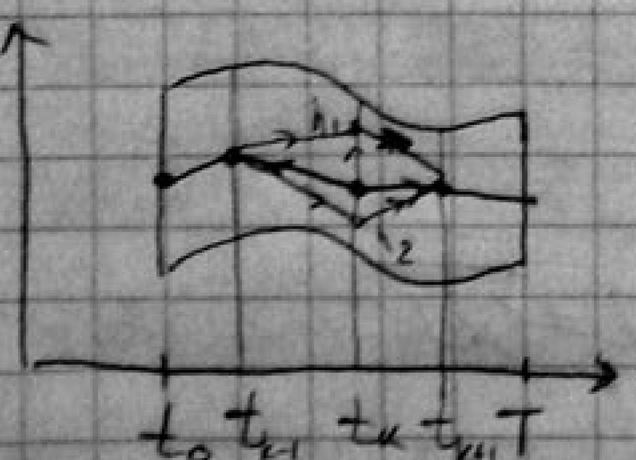
$[t_0, T]$, N-число разбиений, $\tau = \frac{T - t_0}{N}$

$t_k = t_0 + k\tau$ $(N+1)n$ должны хранить в памяти

$x(\tau_k)$

$x_0(t_k)$

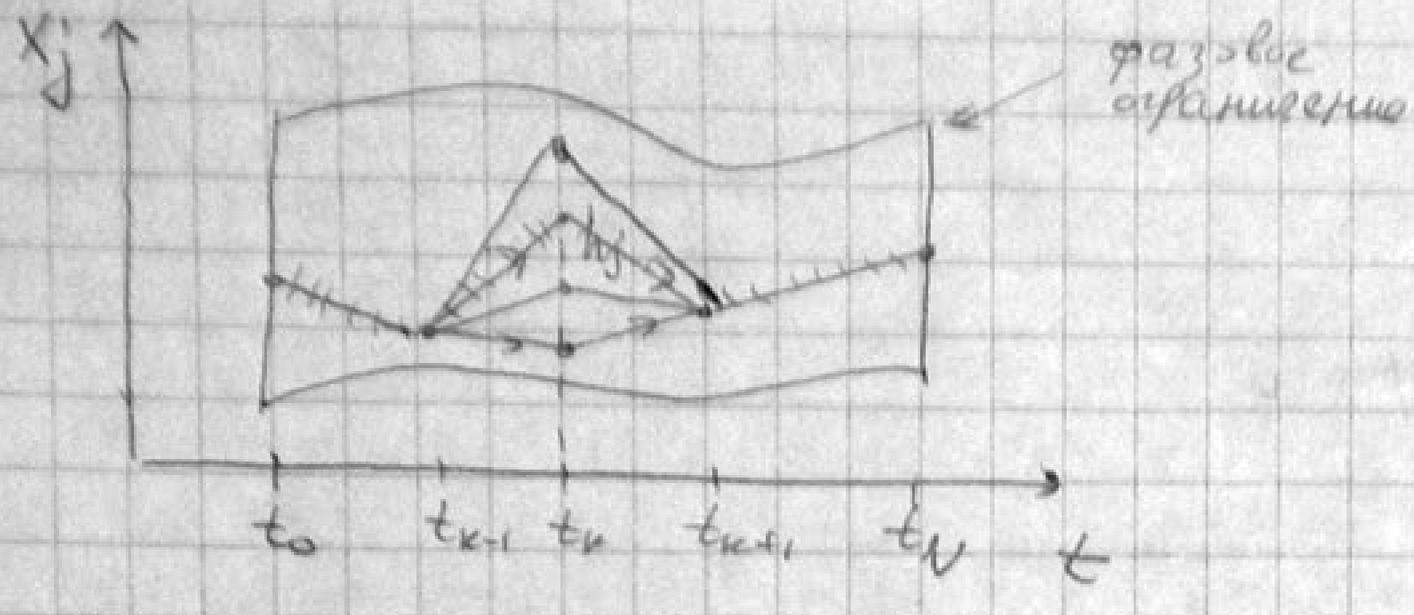




1) $x_0(t_k) \in G(t_k)$

2) t_k, $(t_{k-1}, x_0(t_{k-1}))$, $(t_k, x_0(t_k))$ | $(t_k, x_0(t_k))$

$(t_{k+1}, x_0(t_{k+1}))$



$$\frac{h_j}{\tau^2} \to 0, \ \tau \to 0$$

26.03.09

§ 4. ε-метод Балакришна

$$(1) \ \begin{cases} \dot{x} = f(t, x, u), & [0, T] \\ x(0) = x_0 \\ x(T) = X_T \end{cases}$$

~~$u(t) \in U$~~

$$(2) \quad J(u) = \int_0^T f_0(x(t), u(t), t)\,dt$$

1) $u_0(t) \to u_1(t) \to \dots$
2) $x_0(t) \to x_1(t) \to \dots$

$$(3) \quad J_\varepsilon(u, x) = J(u) + \frac{1}{\varepsilon}\int_0^T |\dot{x} - f(t, x, u)|\,dt \to J_\varepsilon(u, x) = \Phi(\alpha)$$

$u(\cdot), x(\cdot)$, $x(0) = 0$
$x(T) = X_T$

$$(4) \quad u(t) = \sum_{k=0}^{p} \left(a_k \sin\frac{k\pi t}{T} \pm b_k \cos\frac{k\pi t}{T}\right)$$

$$x(t) = x_0 + \frac{t}{T}(x_p - x_0) + \sum_{k=1}^{q} c_k \sin\frac{k\pi t}{T}$$

$$\alpha = (a_k, b_k, c_k) \quad \underbrace{2pm + qn}$$

$[0, T]$ $t_1, t_2, \dots t_N$

$$h_i(\alpha) = \dot{x}(t_i) - f(t_i, x_i(t_i), u(t_i)) \qquad i = \overline{1, N}$$

$$\Phi(\alpha) = J(u) + \frac{1}{\varepsilon}$$

$$h_i(\alpha) = \dot{x}(t_i) - f(t_i, x(t_i), u(t_i)) \qquad i = \overline{1, N}$$

$$\tilde{\Phi}(\alpha) = \tilde{J}(\alpha) + \frac{1}{\varepsilon}\sum_{i=1}^{N} |h_i(\alpha)|^2$$

1. $\textcircled{\alpha}$, $\tilde{J}(u)$, $h_i(\alpha)$

2. $z(\alpha) = (J, h_1, \ldots, h_N)$

$$\frac{\partial z}{\partial \alpha} = \left\{ \frac{\partial J}{\partial \alpha}, \frac{\partial h_1}{\partial \alpha}, \ldots, \frac{\partial h_N}{\partial \alpha} \right\}$$

3. $$\min_{\delta\alpha} \left(J(\alpha) + \frac{\partial J}{\partial \alpha}\delta\alpha + \frac{1}{\varepsilon}\sum_{i=1}^{N} | h_i(\alpha) + \frac{\partial h_i}{\partial \alpha}\delta\alpha |^2 \right)$$

$$\frac{\partial J}{\partial \alpha_j} + \frac{2}{\varepsilon}\sum_{i=1}^{N} \left(h_i + \frac{\partial h_i}{\partial \alpha_j}\delta\alpha_j , \frac{\partial h_i}{\partial \alpha_j} \right) = 0 \to \delta\alpha_j$$

4. $\alpha := \alpha + \delta\alpha$

§5 Элементы вып. анализа

__Опр__ $J(u)$ дифф. в гильбертовом пр-ве , $J'(u) \in H$

$$J(u+h) - J(u) = \langle J'(u), h \rangle_H + o(h, u)$$

$$\frac{|o(h,u)|}{|h|} \to 0 , \quad |h| \to 0$$

__Пример 1__ $J(u) = \langle u, u \rangle$, $J'(u) = 2u$, $J(x) = x^2$, $J'(x) = 2x$

$$J(u+h) - J(u) = \langle u+h, u+h \rangle - \langle u, u \rangle = \langle u, u \rangle + \langle u, h \rangle + \langle h, u \rangle +$$

$$\langle h, h \rangle - \langle u, u \rangle = \langle 2u, h \rangle + |h|$$

$$J'(u) = 2u$$

__Пример 2__

$A : H \to H$, $J(u) = \frac{1}{2}\langle Au, u \rangle - \langle b, u \rangle$

$J'(x) = Ax - b$

$$J(u+h) - J(u) = \frac{1}{2}\langle A(u+h), u+h \rangle - \langle b, u+h \rangle - \frac{1}{2}\langle Au, u \rangle + \langle b, u \rangle$$

$$= \cancel{\frac{1}{2}\langle Au, u \rangle} + \frac{1}{2}\langle Au, h \rangle + \frac{1}{2}\langle Ah, u \rangle + \frac{1}{2}\langle Ah, h \rangle - \cancel{\langle b, u \rangle} - \langle b, h \rangle .$$

$$- \cancel{\frac{1}{2}\langle Au, u \rangle} + \cancel{\langle b, u \rangle} = \frac{1}{2}\langle Au, h \rangle + \frac{1}{2}\langle h, A^* u \rangle - \langle b, h \rangle + \frac{1}{2}\langle Ah, h \rangle =$$

$$\langle Ah, u \rangle = \langle h, A^* u \rangle$$

$$= \left\langle \frac{A + A^*}{2} u, h \right\rangle - \langle b, h \rangle + \frac{1}{2}\langle Ah, h \rangle = \left\langle \left(\frac{A + A^*}{2} u - b \right), h \right\rangle + \underbrace{\frac{1}{2}\langle Ah, h \rangle}_{\substack{\downarrow h \to 0 \\ 0}}$$

$$J'(u) = \frac{A + A^*}{2} u - b$$

если $A = A^* \Rightarrow J'(u) = Au - b$

2.04.09

(2) $J(u) = |x(T,u) - y|^2 \to \inf\limits_u$

$$(1)\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) & [t_0, T] \\ x(t_0) = x_0 \\ \cancel{u(t) \in U} \\ u(t) \in L_2^z[t_0, T] \end{cases}$$

$x - n,\ A\ n \times n\ ,\ u - 2,\ B\ n \times z$

$L_2[t_0, T)$

$f_1(x), f_2(x)$

$$\big(f_1(x), f_2(x)\big)_{L_2^z[t_0,T]} = \int_{t_0}^{T} \big(f_1(x) \cdot f_2(x)\big)_{L_2^z} dx$$

$$|f|_{L^2[t_0,T]} = \sqrt{(f,f)_{L_2^z[t_0,T]}}$$

$f(x) \in L_2^z[t_0, T]$

$$f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_z(x) \end{pmatrix} \qquad f_i(x) \in L_2[t_0, T]$$

$$|f| = \max_{[t_0, T]} f(x)$$

$$\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

$x(t) =$

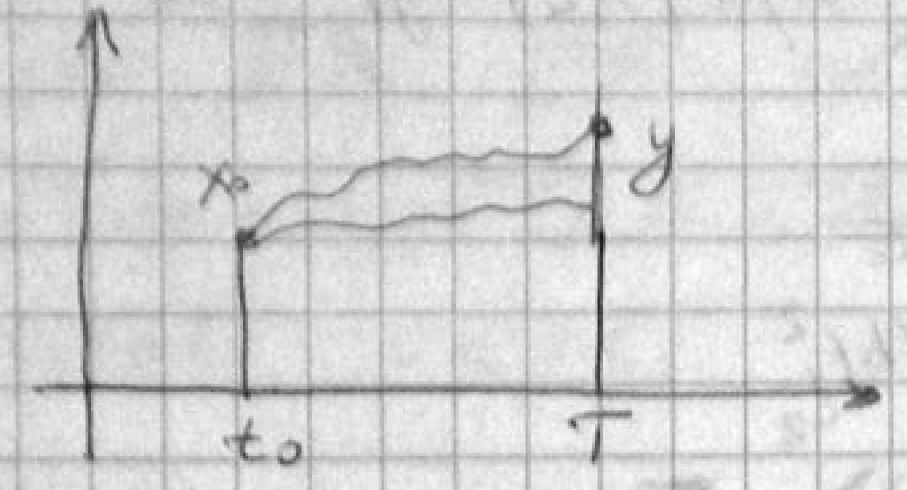
Опр $\quad u(t) \in L_2^r[t_0, T]$

$x(t, u)$

$$x(t, u) = x_0 + \int_{t_0}^{t} \big(Ax(s,u) + Bu(s)\big) ds$$

$$\dot{x}(t, u) = Ax(t, u) + Bu(t)$$

Теорема   u - выпукло замкн. огр → (1)-(2)



Теорема   (1) $u(t) \in L_2^2[t_0, T]$

$$J'(u) = B^* \psi(t,u) \, , \ [t_0, T]$$

$$(3) \begin{cases} \dot{\psi}(t,u) = -A^* \psi(t,u) \, , \ [t_0, T] \\ \psi(T,u) = 2\left(x(T,u) - y\right) \end{cases}$$

$$u_0(t) \xrightarrow{(2)} x(t,u_0) \xrightarrow{(3)} \psi(t,u_0) - J'(u_0) = B^* \psi(t,u_0)$$

Док-во:

$u, u+h \in L_2^2[t_0, T]$

$x(t,u), \ x(t,u+h)$

$\Delta x(t) = x(t,u+h) - x(t)$

$$- \begin{array}{l} \dot{x}(t,u+h) = A x(t,u+h) + B(u(t) + h(t)) \qquad x(t_0, u+h) = x_0 \\ \dot{x}(t,u) = A x(t,u) + B u(t) \end{array}$$

$$\begin{cases} \Delta \dot{x}(t) = A \Delta x(t) + B h(t) \qquad [t_0, T] \\ \Delta x(t_0) = 0 \end{cases}$$

$$\Delta x(t) = 0 + \int_{t_0}^{t} (A x(\tau) + B h(\tau)) \, d\tau$$

$$|\Delta x(t)| \le |A| \int_{t_0}^{t} |x(\tau)| \, d\tau + |B| \int_{t_0}^{t} |h(\tau)| \, d\tau$$

Условия Леммы Гронуолла-Белмана вып.

$\varphi(t), b(t) \ge 0 \qquad [t_0, T]$

$$\varphi(t) \le a \int_{t_0}^{t} \varphi(\tau) \, d\tau + b(t)$$

$$|\Delta x(t)| \le |B| \, e^{|A|(T - t_0)} \int_{t_0}^{T} |h(\tau)| \, d\tau$$

$$\int_{t_0}^{T} 1 \cdot |h(\tau)| \, d\tau \le (T - t_0)^{1/2} \left(\int_{t_0}^{T} |h(\tau)|^2 dt \right)^{1/2} \Rightarrow |\Delta x(t)| \le C_1 |h|_{L_2}$$

$$I(u+h) - I(u) = |x(T,u) + \Delta x(T) - y|^2_{E^n} - |x(T,u) - y|^2_{E^n} =$$

$$= 2 \langle x(T,u) - y, \Delta x(T) \rangle_{E^n} + \underbrace{|\Delta x(T)|^2_{E^n}}_{\leq c_1 |h|^2_{L_2}}$$

$$2 \langle x(T,u) - y, \Delta x(T) \rangle = \langle \Psi(T,u), \Delta x(T) \rangle \underset{\text{формуле Ньютона-Лейбница}}{=}$$

$$= \int_{t_0}^{T} \frac{d}{dt} \langle \Psi(t,u), \Delta x(t) \rangle dt + \langle \Psi(t_0,u), \underset{0}{\underset{\parallel}{\Delta x(t_0,u)}} \rangle =$$

$$= \int_{t_0}^{T} \left(\langle \psi, \Delta \dot{x} \rangle + \langle \dot{\psi}, \Delta x \rangle \right) dt = \int_{t_1}^{T} \left(\langle \psi, A\Delta x + Bh \rangle + \right.$$
$$\left. + \langle -A^* \psi, \Delta x \rangle \right) dt \;\textcircled{=}$$

$$\psi(T) = 2(x(T,u) - y)$$
$$\dot{\psi} = -A^* \psi$$

$$\textcircled{=} \int_{t_0}^{T} \langle B^* \psi, h \rangle + \langle \psi, A\Delta x \rangle - \langle A^* \psi, \Delta x \rangle) dt =$$

$$= \int_{t_0}^{T} \langle B^* \psi(t,u), h(t) \rangle_{E^n} dt$$

$$I(u+h) - I(u) = \int_{t_0}^{T} \langle B^* \psi(t,u), h(t) \rangle dt + \underbrace{|\Delta x(\overset{t}{T})|^2}_{< C_1 |h|^2_{L_2}}$$

16.04.09

ф-ие

Опр1 $I(u)$ вып. на $U \subset H$ (U - вып) если

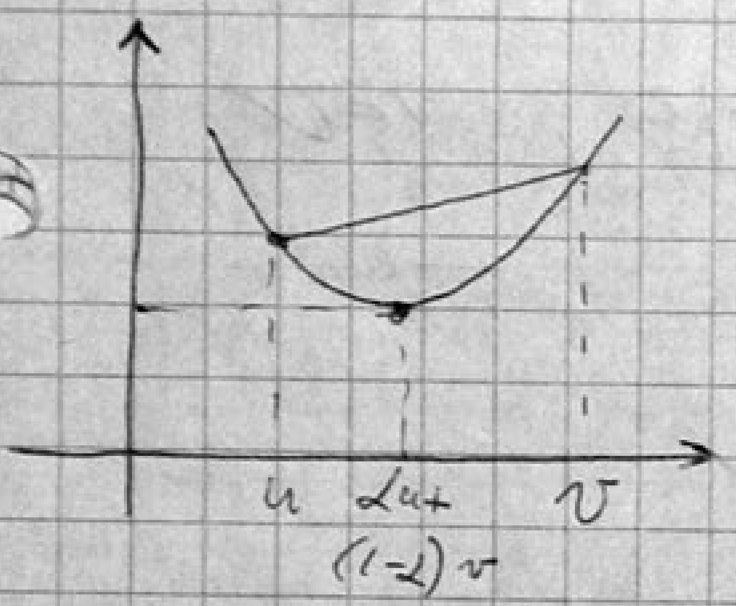
$$I(\alpha u + (1-\alpha) v) \le \alpha I(u) + (1-\alpha) I(v)$$

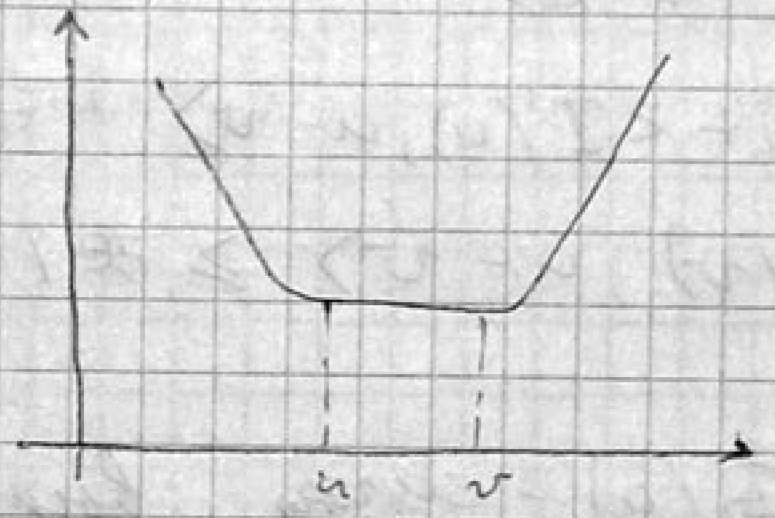
$u, v \in U$, $0 \le \alpha \le 1$

Опр2 $\alpha \ne 0, \alpha \ne 1$

Ⓟ $I(u) = \langle u,u \rangle = \|u\|^2$

$I(u) = \langle c,u \rangle$





Опр3 $I(u)$ сильно вып., $u \in U \subset H$

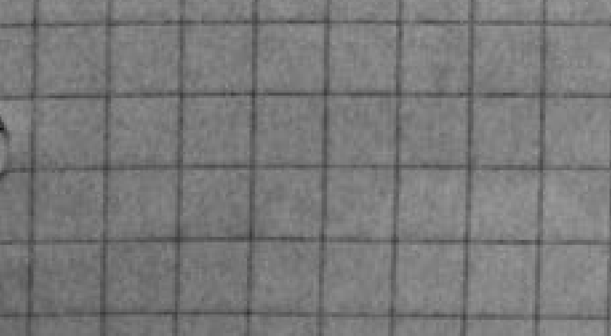
$$I(\alpha u + (1-\alpha) v) \le \alpha I(u) + (1-\alpha) I(v) - \varkappa \alpha (1-\alpha) |u-v|_H^2 \le$$
$$\le \alpha I(u) + (1-\alpha) I(v)$$

Ⓟ $I(u) = \langle u,u \rangle$

$f(x) = |x|$ – строго выпукла, но не сильно

Св-ва:

1° $I(u)$ – вып $\Rightarrow$ $I(u)$ непр во всех внутр. точках



← вып, но не непр.

2° $I(u)$ строго вып.

Теор V - вып., $I(u) \in C^1(V)$

$I(u)$ - выпукла $\Leftrightarrow$

1) $I(u) \geq I(u) + \langle I'(u), u - v \rangle$, $u, v \in V$

2) $\langle I'(u) - I'(v), u - v \rangle \geq 0$, —''—

$f'(u) - f'(v) \cdot (u - v) \geq 0$

$u \geq v$, $f'(u) \geq f'(v)$

Теор2 V - вып., $I(u) \in C^1(V$

$I(u)$ - сильно вып $\Leftrightarrow$

1) $I(u) \geq I(u) + \langle I'(u), u - v \rangle + \varkappa |u - v|^2$, $u, v \in V$

2) $\langle I'(u) - I'(v), u - v \rangle \geq \varkappa |u - v|^2$ —''—

(П) $I(u)$ - вып., $I_2(u)$ - сильно вып. $\Rightarrow$

$\Rightarrow I_1(u) + I_2(u)$ - сильно вып.

$$I_1(\alpha u + (1-\alpha) v) \leq \alpha I_1(u) + (1-\alpha) I_1(v)$$

$\oplus$

$$I_2(\alpha u + (1-\alpha) v) \leq \alpha I_2(u) + (1-\alpha) I_2(v) - \varkappa \alpha (1-\alpha) |u - v|^2$$

$$I_1(\alpha u + (1-\alpha) v) + I_2(\alpha u + (1-\alpha) v) \leq \alpha (I_1(u) + I_2(u)) + (1-\alpha)(I_1(v) + I_2(v)) - \varkappa \alpha (1-\alpha) |u - v|^2$$

Это означает, что $I_1(u) + I_2(u)$ - сильно вып.

$$I(u) = |x(T, u) - y|^2 + \varepsilon \int_{t_0}^{T} |u(t)|^2_{E^2} dt$$

$$\begin{cases} \dot{x} = Ax + Bu \\ x(t_0) = x_0 \\ u \in V \subset L_2^2 [t_0, T] \end{cases}$$

§6 Метод скорейшего спуска

$u_{k+1} = u_k - \alpha_k I'(u_k) \qquad k = 0, 1 \ldots$

$u_1 = u_0 - \alpha_0 I'(u_0)$

1) $I'(u_k) = 0$

2) $I'(u_k) \neq 0$

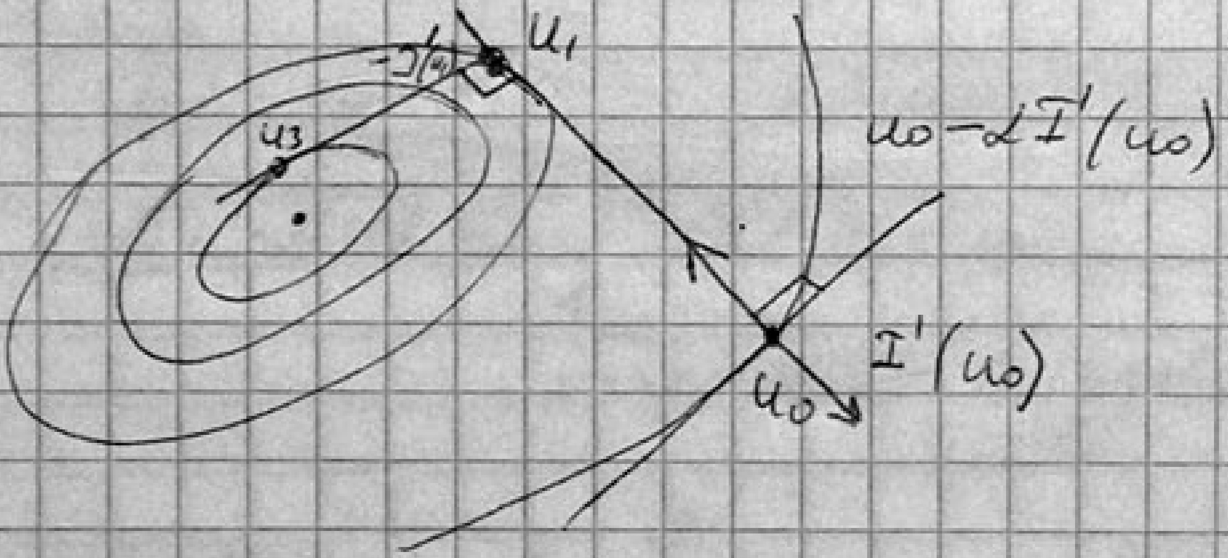
$\alpha_k : I(u_{k+1}) < I(u_k)$

I. $\min\limits_{\alpha \geq 0} I(u_k - \alpha I'(u_k)) = I(u_k - \alpha_k I'(u_k))$

II $\alpha_k : I(u_{k+1}) < I(u_k)$

$|I(u_{k+1}) - I(u_k)| \leq \varepsilon$

Линии уровня $\{u \in U : I(u) \leq C\}$



Метод скорейшего спуска сходится быстро

$I(u) = |x(T, u) - y|^2$

$\dot{x}(t) = Ax(t) + Bu(t)$

$x(t_0) = x_0$

$u(t) \in U = L_2^2[t_0, T]$

$I'(u) = B^* \psi(t, u)$

$\dot{\psi}(t, u) = -A^* \psi(t, u)$

$\psi(T, u) = 2(x(T, u) - y)$

$u_{k+1}(t) = u_k(t) - \alpha_k \cdot B^* \psi(t, u_k)$

$g_k(\alpha) = |x(T, u_k - \alpha B^* \psi(t, u_k)) - y|^2$

$$g_k(\alpha) = J(u_k - \alpha B^* \psi(t, u_k)) = J(u_k) - \alpha |J'(u_k)|^2 +$$
$$+ \alpha^2 |x(T, u_k - J'(u_k)) - x(T, u_k)|^2 \to \min_\alpha$$

$$\alpha_k = \frac{|J'(u_k)|^2}{2|x(T, u_k - J'(u_k)) - x(T, u_k)|^2} \geq 0$$

$$u_0 \xrightarrow{①} x(t, u_0) \xrightarrow{②} \psi(t, u_0) \to J'(u_0) \to \overset{③}{x(T, u_0 - J'(u_0))}$$

$$\alpha_0$$
$$\downarrow$$
$$u_1$$

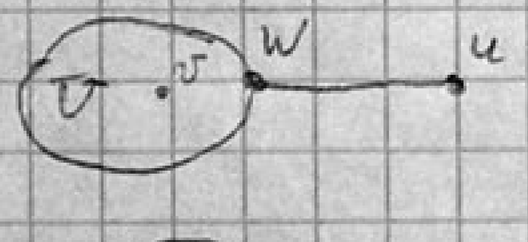
23.04.09

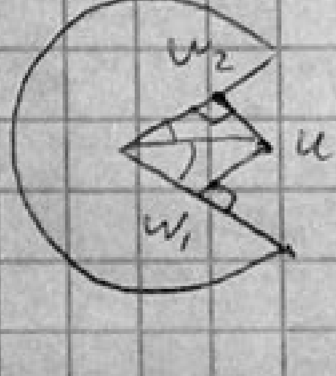
§6 МУГ

$J(u) \to \inf, u \in U$

Опр 1 $u \in H$, $w = P_U(u)$

$$|u - w| = \inf_{v \in U} |u - v|$$

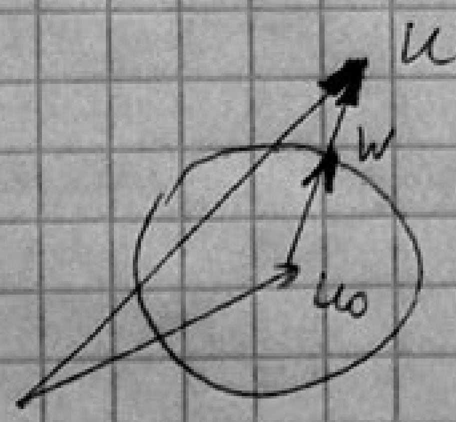




Теор U - вып., замкн., H

$\Rightarrow u \in H \ \exists! w$

Пример 1 $U = \{u \in H \ / \ |u - u_0| \leq R\}$

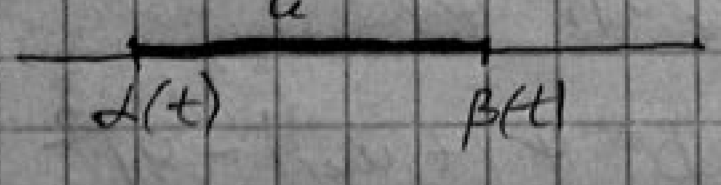


$$P_U(u) = \begin{cases} u, & |u - u_0| \leq R \\ u_0 + \frac{u - u_0}{|u - u_0|} \cdot R, & |u - u_0| > R \end{cases}$$

Пример 2 $U = \{ u \in L_2^r[t_0, T] \ / \ \alpha_i(t) \leq u_i(t) \leq \beta_i(t),$

почти всюду $[t_0, T]$

$i = \overline{1, r}$



$$P_U(u) = \begin{cases} \alpha(t), & u_i(t) < \alpha_i(t) \\ u(t), & \alpha_i(t) \le u_i(t) \le \beta_i(t) \\ \beta(t), & \alpha_i(t) > \beta_i(t) \end{cases}$$

Теор] U - вып., замкн., ~~непр~~ непр. $U_* \neq \varnothing$

$J(u)$ - в, $J(u) \in C^1(H)$

$u_* \in U_* \Leftrightarrow u_* = P_U(u_* - \alpha J'(u_*)) \qquad \alpha > 0$

$u_* = u_* - \alpha J'(u_*)$

$J'(u_*) = 0$

$\inf J(u), \ u \in U$

минимизир. посл-ть $\quad u_{k+1} = P_U(u_k - \alpha_k J'(u_k))$

1) $u_{k+1} = u_k \Rightarrow u_k \in U_*$

2)

Ⅰ $\min J(P_U(u_k - \alpha J'(u_k))) = J(P_U(u_k - \alpha_k J'(u_k)))$

Ⅱ $\alpha_k = \alpha > 0, \quad J(u_{k+1}) < J(u_k)$

Ⅲ $J(u) \in C^{1,1}(U)$

$\uparrow$

непр.-дифф и удовл. усл. Липшица

$0 < \varepsilon_0 \le \alpha_k \le \dfrac{2}{L + 2\varepsilon_0}$ $\qquad \varepsilon_0$ - пар-р данного метода

Теор U - вып. замкн. в пр-ве H, $J(u) \in C^{1,1}(U)$

$u_k \to u_*, \quad |u_k - u_*| \le C_1 q^k, \quad 0 < q < 1$

$J(u) = |x(T, u) - y|^2 \to \inf$

$\dot{x}(t) = Ax(t) + Bu(t) \qquad [t_0, T]$

$x(t_0) = x_0$

$u(t) \in U = \{u(t) = (u_1(t), \dots, u_r(t))$

$u_i(t) \in L_2[t_0, T], \quad \alpha_i(t) \le u(t) \le \beta_i(t)\}$

$$u_{k+1}(t) = P_U(u_k(t) - \alpha_k \cdot J'(u_k))$$

$$J'(u) = B^* \psi(t, u)$$

$$\psi(t, u) = -A^* \psi(t, u)$$

$$\psi(T, u) = 2(x(T, u) - y)$$

$$u_{k+1}^i(t) = \begin{cases} \to \alpha_i(t), & u_k^i(t) - \alpha_k (B^* \psi(t, u_k))^i < \alpha_i(t) \\ \to u_k^i(t) - \alpha_k (B^* \psi(t, u_k))^i, & \alpha_i(t) \le u_k^i(t) - \alpha_k \cdot (B^* \psi(t, u_k))^i \\ \to \beta_i(t), & u_k^i(t) - \alpha_k (B^* \psi(t, u_k))^i > \beta_i(t) \end{cases}$$

$$J(u_{k+1}) < J(u_k)$$

$$u(t) \in U_R = \{u(t) \in L_2^r[t_0, T], \int_0^T |u(t) - \bar{u}(t)| dt \le R^2\}$$

$$u_{k+1} = \begin{cases} \bar{u}(t) + R \dfrac{u_k(t) - \alpha_k B^* \psi(t, u_k) - \bar{u}(t)}{(\underbrace{\int_{t_0}^T |u_k(t) - \alpha_k B^* \psi(t, u_k) - \bar{u}(t)|_{E^2}^2 dt}_{A})^{1/2}}, & A > R^2 \\ u_k(t) - \alpha_k \cdot B^* \cdot \psi(t, u_k), & A < R^2 \end{cases}$$

§7 $J(u) \to \inf, u \in U$ –

$J(u) \in C^1(U)$

$u_0 \in U$

$\overline{u_k} \in U$

$$\min_{u \in U} \langle J'(u_k), u - u_k \rangle_H = \langle J'(u_k), \overline{u_k} - u_k \rangle$$

$u_k = \overline{u_k}$

$u_{k+1} = u_k + \alpha_k (\bar{u}_k - u_k)$

$0 \le \alpha_k \le 1$

① $\min\limits_{0 \le \alpha \le 1} J(u_k + \alpha(\bar{u}_k - u_k)) = J(u_k + \alpha_k(\bar{u}_k - u_k))$

② $\alpha_k = \alpha \qquad J(u_{k+1}) < J(u_k)$

Теор

$\boxed{\text{14 мая – зачет}}$

$\boxed{\text{30 апр. – не будет}}$

7 мая – послед. лекция

7.05.09

Разностная апроксимация ЗОУ

$$\begin{cases} \dot{x}(t) = Ax(t) + u(t) \qquad [t_0, T] \\ x(t_0) = x_0 \\ u(t) \in U \\ I(u) = |x(T,u) - y|^2 \to \inf \end{cases}$$

$u(t)$ – задано

$u^{(m+1)}(t) = P_U(u^m(t) - \alpha^m I'(u^m(t))$

$\alpha^m > 0, \quad m = 0, 1, \ldots$

$P_U(u) = W$

$\alpha^m: I(n^{m+1}) < I(u^m)$

$[t_0, N] \quad N$

$t_{k+1} = t_k + h$, $k = \overline{0, N-1}$

$h = \frac{T - t_0}{N}$

① $u^0(t_k) = u_k^0$ $k = \overline{0, N-1}$

$x^0(t_k) = x_k^0$

$$\begin{cases} x_{k+1}^0 = x_k^0 + h(Ax_k^0 + u_k^0) \\ x_0^0 = x_0 \end{cases}$$

$I(u_0) = |x_N^0 - y|^2$

$u_k^0, x_k^0, I(u^0), \alpha^0$

② $\alpha^m, u_k^m, x_k^m, I(u^m)$

а) $\dot{\psi} = -A^*\psi$, $\Psi(T) = 2(x(T,u) - y)$, $I'(u) = \Psi(t)$

$$\begin{cases} \Psi_{k+1}^m = \Psi_k^m - hA^*\Psi_k^m \\ \Psi_N^m = 2(x_N^m - y) \end{cases}$$

$\Psi_N^m = \Psi_{N-1}^m - hA^*\Psi_{N-1}^m$

$\Psi_N^m = (E - hA^*)\Psi_{N-1}^m$

$\Psi_{N-1}^m = (E - hA^*)^{-1}\Psi_N^m$

б) $u_k^{m+1} = P_U(u_k^m - \alpha^m \Psi_k^m)$, $k = \overline{0, N-1}$

в) $\begin{cases} x_{k+1}^{m+1} = x_k^{m+1} + h(Ax_k^{m+1} + u_k^{m+1}) \\ x_0^{m+1} = x_0 \end{cases}$ ← Методом Рунге-Кутта

г) $I(u^{m+1}) = |x_N^{m+1} - y|^2$

③ $I(u^{m+1}) < I(u^m)$

Если "да" $\alpha^{m+1} := \alpha^m$ Переходим к ②

Если "нет" $\alpha^m := \frac{1}{2}\alpha^m$ Переходим к ⓑ

То же самое можно сделать Методом Условного Градиента