766, 769, 773, 775, 757, 759, 761

D.p/

№ 766

$$\begin{cases} y'' + y' = f(x) \\ y(0) = 0 \\ y'(1) = 0 \end{cases}$$

1) $\lambda^2 + \lambda = 0 \Rightarrow \lambda = 0, -1$

$y(x) = C_1 + C_2 e^{-x}$

$C_1 + C_2 = 0 \qquad -C_2 e^{-1} = 0$

2) $y_1: C_1 = -C_2 \quad C_2 \neq 0 \quad y_1(x) = 1 - e^{-x}$

$y_2: C_1 \neq -C_2 \quad C_2 = 0 \quad y_2(x) = 1$

3) $G(x,s) = \begin{cases} a(s)(1-e^{-x}), & 0 \le x \le s \\ b(x) & s < x \le 1 \end{cases}$

$$\begin{cases} a(s)(1-e^{-s}) = b(s) \\ a(s)e^{-s} + 1 = 0 \end{cases} \Leftrightarrow \begin{cases} a(s) = -e^{s} \\ b(s) = -e^{s} + 1 \end{cases}$$

$G(x,s) = \begin{cases} e^{s}(e^{-x} - 1) & , 0 \le x \le s \\ 1 - e^{s} & , s < x < 1 \end{cases}$

Ответ: $G(x,s) = \begin{cases} e^{s}(e^{-x} - 1), & 0 \le x \le s \\ 1 - e^{s}, & s < x < 1 \end{cases}$

№ 769

$$\begin{cases} x^2 y'' + 2xy' = f(x) \\ y(1) = 0 \\ y'(3) = 0 \end{cases}$$

1) $\lambda(\lambda - 1) + 2\lambda = 0 \Leftrightarrow \lambda^2 + \lambda = 0 \Rightarrow \lambda = 0, -1$

$y(t) = C_1 + C_2 e^{-t}$

$y(x) = C_1 + C_2 x^{-1}$

2) $y(1) = C_1 + C_2 \qquad y'(3) = -\frac{C_2}{9} = 0$

$y_1: C_1 + C_2 = 0 \quad C_2 \neq 0 \quad y_1 = 1 - \frac{1}{x}$

$y_2: C_1 \neq -C_2 \quad C_2 = 0 \quad y_2 = 1$

3) $G(x,s) = \begin{cases} a(s)(1 - \frac{1}{x}), & 1 \le x \le s \\ b(s) & , s < x < 3 \end{cases}$

$$\begin{cases} a(s)(1 - \frac{1}{s}) = b(s) \\ 0 = a(s)\frac{1}{s^2} + \frac{1}{s^2} \end{cases} \Leftrightarrow \begin{cases} a(s) = -1 \\ b(s) = \frac{1}{s} - 1 \end{cases}$$

Ответ: $G(x,s) = \begin{cases} \frac{1}{x} - 1, & 1 \le x \le s \\ \frac{1}{s} - 1, & s < x \le 3 \end{cases}$

№ 773

$$\begin{cases} y'' + y' = f(x) \\ y'(0) = 0 \\ y(+\infty) = 0 \end{cases}$$

1) $\lambda^2 + \lambda = 0 \Rightarrow \lambda = 0; -1$

$y(x) = C_1 + C_2 e^{-x}$

$y_1: \; C_2 = 0 \quad C_1 \neq 0 \qquad y_1 = 1$

$y_2: \; C_2 \neq 0 \quad C_1 = 0 \qquad y_2 = e^{-x}$

3) $G(x,s) = \begin{cases} a(s), & 0 \le x \le s \\ b(s)e^{-x}, & s < x < +\infty \end{cases}$

$\begin{cases} a(s) = b(s)e^{-s} \\ -b(s)e^{-s} = 1 \end{cases} \iff \begin{cases} a(s) = -1 \\ b(s) = -e^{s} \end{cases}$

Ответ: $G(x,s) = \begin{cases} -1, & 0 \le x \le s \\ -e^{s-x}, & s < x < +\infty \end{cases}$

№775

$\begin{cases} y'' + 4y' + 3y = f(x) \\ y(0) = 0 \\ y(x) = O(e^{-2x}), & x \to +\infty \end{cases}$

1) $\lambda^2 + 4\lambda + 3 = 0 \Rightarrow \lambda = -1; -3$

$y(x) = C_1 e^{-x} + C_2 e^{-3x}$

2) $C_1 + C_2 = 0 \qquad C_1 = 0$

$y_1: \; C_1 = -C_2 \quad C_1 \neq 0 \qquad y_1 = e^{-x} - e^{-3x}$

$y_2: \; C_1 \neq -C_2 \quad C_1 = 0 \qquad y_2 = e^{-3x}$

3) $G(x,s) = \begin{cases} a(s)(e^{-x} - e^{-3x}), & 0 \le x \le s \\ b(s)e^{-3x}, & s < x < +\infty \end{cases}$

$-3b(s)e^{-3s} = a(s)(3e^{-3s} - e^{-s}) + 1$

$\iff \begin{cases} a(s)(e^{2s} - 1) = b(s) \\ -3b(s) = a(s)(3 - e^{2s}) + e^{3s} \end{cases} \iff \begin{cases} b(s) = a(s)(e^{2s} - 1) \\ a(s)(3 - e^{2s} + 3e^{2s} - 3) + e^{3s} = 0 \end{cases} \iff$

$\iff \begin{cases} a(s) = -\frac{e^{s}}{2} \\ b(s) = (1 - e^{2s})\frac{e^{s}}{2} \end{cases}$

Ответ: $G(x,s) = \begin{cases} \frac{e^{s}}{2}(e^{-3x} - e^{-x}), & 0 \le x \le s \\ \frac{e^{s}}{2}(1 - e^{2s})e^{-3x}, & s < x < +\infty \end{cases}$

№757

$\begin{cases} y'' - y' - 2y = 0 \\ y'(0) = 2 \\ y(+\infty) = 0 \end{cases}$

$y(x) = u(x) - 2e^{-x} \Rightarrow u''(x) - u'(x) - 2u(x) = 0$

1) $\lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda = 2; -1$

$y(x) = C_1 e^{2x} + C_2 e^{-x} \Rightarrow u(x) = C_1 e^{2x} + C_2 e^{-x}$

2) $2C_1 - C_2 = 0 \qquad C_1 = 0$

$y_1: \; 2C_1 = C_2 \quad C_1 \neq 0 \qquad y_1 = e^{2x} + 2e^{-x}$

$y_2: \; 2C_1 \neq C_2 \quad C_1 = 0 \qquad y_2 = e^{-x}$

3) $G(x,s) = \begin{cases} a(s)(e^{2x} + 2e^{-x}), & 0 \le x \le s \\ b(s)e^{-x}, & s < x < +\infty \end{cases}$

$\begin{cases} a(s)(e^{2s} + 2e^{-s}) = b(s)e^{-s} \\ -b(s)e^{-s} = a(s)(2e^{2s} - 2e^{-s}) + 1 \end{cases} \iff \begin{cases} b(s) = a(s)(e^{3s} + 2) \\ a(s)(2e^{2s} - \cancel{2e^{-s}} + e^{2s} + \cancel{2e^{-s}}) = -1 \end{cases} \iff$

$\iff \begin{cases} a(s) = -\frac{1}{3}e^{-2s} \\ \ldots \end{cases} \Rightarrow$

$$G(x,s) = \begin{cases} \ldots & , 0 \le x \le s \\ -\frac{1}{3}(e^{s}+2e^{-2s})e^{-x} & , s < x < +\infty \end{cases}$$

$$u(x) = \int_0^{+\infty} G(x,s)\cdot 0\, ds = 0 \Rightarrow y(x) = u(x) - 2e^{-x} = -2e^{-x}$$

Ответ: $y(x) = -2e^{-x}$

№ 759

$$\begin{cases} y'' - 2iy = 0 \\ y(0) = -1 \\ y(+\infty) = 0 \end{cases}$$

$$y(x) = u(x) + e^{-x}(i\sin x - \cos x)$$

$$y'(x) = u'(x) - e^{-x}(i\sin x - \cos x) + e^{-x}(i\cos x + \sin x) =$$
$$= u'(x) + e^{-x}(i(\cos x - \sin x) + \sin x + \cos x)$$

$$y''(x) = u''(x) - e^{-x}(i(\cos x - \sin x) + \sin x + \cos x) +$$
$$+ e^{-x}(i(-\sin x - \cos x) + \cos x - \sin x) =$$
$$= u''(x) + e^{-x}(-2i\cos x - 2\sin x) \Rightarrow$$

$$u'' - 2iu = e^{-x}(-2i\cos x - 2\sin x + 2\sin x + 2i\cos x) = 0$$

1) $\lambda^2 - 2i = 0 \Rightarrow \lambda = \pm 1 \pm i$

$$u(x) = C_1 e^{x}(\cos x + i\sin x) + C_2 e^{-x}(\cos x - i\sin x)$$

2) $C_1 + C_2 = 0$ $\qquad C_1 = 0$

$y_1:$ $C_1 = -C_2$ $\quad C_1 \ne 0$ $\qquad y_1 = e^{x}(\cos x + i\sin x) - e^{-x}(\cos x - i\sin x)$

$y_2:$ $C_1 \ne -C_2$ $\quad C_1 = 0$ $\qquad y_2 = e^{-x}(\cos x - i\sin x)$

$$G(x,s) = \begin{cases} a(s)[\cos x(e^{x}-e^{-x}) + i\sin x(e^{x}+e^{-x})] & , 0 \le x \le s \\ b(s)\, e^{-x}(\cos x - i\sin x) & , s < x < +\infty \end{cases}$$

$$\begin{cases} a(s)[\cos s(e^{s}-e^{-s}) + i\sin s(e^{s}+e^{-s})] = b(s)e^{-s}(\cos s - i\sin s) \\ b(s)[-e^{-s}(\cos s - i\sin s) + e^{-s}(-\sin s + i\cos s)] = \\ = a(s)[-\sin s(e^{-s}-e^{s}) + \cos s(e^{s}+e^{-s}) - i\cos s(e^{s}+e^{-s}) + i\sin s(e^{s}-e^{-s})] + 1 \end{cases} \Rightarrow$$

$\cancel{a(s)(e^{s}-e^{-s}) = b(s)e^{-s}}$

$\cancel{a(s)(e^{s}+e^{-s}) = -b(s)e^{-s}}$

$$\begin{cases} b(s) = \dfrac{a(s)[\cos s(e^{s}-e^{-s}) + i\sin s(e^{s}+e^{-s})]}{e^{-s}(\cos s - i\sin s)} \\ 2a(s)\, e^{s}(\cos s + \sin s - i\cos s + i\sin s) = -1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} a(s) = \dfrac{-1}{2e^{s}(\cos s + \sin s - i\cos s + i\sin s)} \\ b(s) = \dfrac{-(\cos s(e^{s}-e^{-s}) + i\sin s(e^{s}+e^{-s}))}{2(\cos s - i\sin s)(\cos s + \sin s - i\cos s + i\sin s)} \end{cases}$$

$$u(x) = \int_0^{+\infty} G(x,s)\cdot 0\cdot ds = 0 \Rightarrow y(x) = e^{-x}(i\sin x - \cos x)$$

Ответ: $y(x) = e^{-x}(i\sin x - \cos x)$

№ 761

$$\begin{cases} x^2 y'' - 2xy' + 2y = 0 \\ y(x) = o(x), \ x \to 0 \\ y(1) = 3 \end{cases}$$

$$y(x) = u(x) + 3x^3$$

$$x^2(u'' + \ldots$$

$\Leftrightarrow x^2u'' - 2xu' + 2u = -6x^3$

$\lambda(\lambda-1) - 2\lambda + 2 = 0 \Leftrightarrow \lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda = 2; 1$

$u(t) = C_1e^{2t} + C_2e^{t} \Rightarrow u(x) = C_1x^2 + C_2x$

2) $C_2 = 0$ $\qquad C_1 + C_2 = 0$

$y_1: \; C_2 = 0 \qquad C_1 \ne -C_2 \qquad y_1 = x^2$

$y_2: \; C_2 \ne 0 \qquad C_1 = -C_2 \qquad y_2 = x^2 - x$

)

$$G(x,s) = \begin{cases} a(s)\,x^2, & 0 \le x \le s \\ b(s)(x^2-x), & s < x < 1 \end{cases}$$

$$\begin{cases} a(s)s^2 = b(s)(s^2-s) \\ b(s)(2s-1) = 2a(s)s + 1 \end{cases} \Leftrightarrow \begin{cases} a(s)s = b(s)(s-1) \\ b(s)(2s-1-2s+2) = \frac{1}{s^2} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} a(s) = \frac{1}{s^2} - \frac{1}{s^3} \\ b(s) = \frac{1}{s^2} \end{cases}$$

$$G(x,s) = \begin{cases} x^2\left(\frac{1}{s^2} - \frac{1}{s^3}\right), & 0 \le x \le s \\ \frac{1}{s^2}(x^2-x), & s < x < 1 \end{cases}$$

$$u(x) = -6\int_0^1 G(x,s)\,s^3\,ds = -6x^2\int_x^1 (s - 1)\,ds - 6(x^2-x)\int_0^x s\,ds =$$

$$\cancel{= -6\left(x^2\cdot\left(\frac{x^4}{4} - \frac{x^3}{2}\right) + (x^2-x)\left(\frac{1}{4} - \frac{x^4}{4}\right)\right) =}$$

$$\cancel{= -\frac{6}{4}\left(x^6 - 2x^5 + x^2 - x \ldots\right)}$$

$$= -\frac{6}{2}\left(x^2(1 - x^2 - 2 + 2x) + x^4 - x^3\right) =$$

$$= -3\left(x^2 - \cancel{x^4} - 2x^2 + 2x^3 + \cancel{x^4} - x^3\right) = 3x^2 - 3x^3 \Rightarrow$$

$y(x) = u(x) + 3x^3 = 3x^2$

Ответ: $y(x) = 3x^2$

Д.р

№1066

$$\begin{cases} y' = y + y^2 + xy^3 \\ y(2) = y_0 \end{cases} \qquad \left.\frac{\partial y}{\partial y_0}\right|_{y_0=0}$$

1) $z = \frac{\partial y}{\partial y_0}$

$$\begin{cases} z' = z + 2yz + 3xy^2z \\ z(2) = 1 \end{cases}$$

2) $y_0 = 0$

$$\begin{cases} y' = y + y^2 + xy^3 \\ y(2) = 0 \end{cases} \qquad \begin{cases} z' = z + 2yz + 3xy^2z \\ z(2) = 1 \end{cases}$$

$y(x) = 0 \Rightarrow z' = z \Leftrightarrow \int\frac{dz}{z} = \int dx \Leftrightarrow z = e^{x+C}$

$z(2) = e^{2+C} = 1 \Rightarrow C = -2 \Rightarrow z(x) = e^{x-2}$

Ответ: $e^{x-2} = \left.\frac{\partial y}{\partial y_0}\right|_{y_0=0}$

№1065

$$\begin{cases} y' = 2x + \mu y^2 \\ y(0) = \mu - 1 \end{cases} \qquad \left.\frac{\partial y}{\partial \mu}\right|_{\mu=0}$$

1) Пусть $z = \frac{\partial y}{\partial \mu}$

$z(0) = 1$

$\mu = 0$

$\begin{cases} y' = 2x \\ y(0) = -1 \end{cases}$ $\begin{cases} z' = y^2 \\ z(0) = 1 \end{cases}$

$y' = 2x \Rightarrow y = x^2 + C, \quad y(0) = C = -1 \Rightarrow y(x) = x^2 - 1$

$z' = x^4 + 1 - 2x^2 \Rightarrow z(x) = \frac{x^5}{5} + x - \frac{2x^3}{3} + C$

$z(0) = C = 1 \Rightarrow z(x) = x - \frac{2x^3}{3} + \frac{x^5}{5} + 1$

Ответ: $\left.\frac{\partial y}{\partial \mu}\right|_{\mu=0} = x + 1 - \frac{2x^3}{3} + \frac{x^5}{5}$

№1068

$\begin{cases} \dot{x} = x^2 + \mu t x^3 \\ x(0) = 1 + \mu \end{cases}$ $\left.\frac{\partial x}{\partial \mu}\right|_{\mu=0} - ?$

Пусть $z = \frac{\partial x}{\partial \mu}$

$\dot{z} = 2xz + tx^3 + 3\mu t x^2 z$

$z(0) = 1$

$\mu = 0$

$\begin{cases} \dot{x} = x^2 \\ x(0) = 1 \end{cases}$ $\begin{cases} \dot{z} = 2xz + tx^3 \\ z(0) = 1 \end{cases}$

$\int \frac{dx}{x^2} = \int dt \Leftrightarrow -\frac{1}{x} = t + C \Rightarrow -\frac{1}{1} = 0 + C = -1 \Rightarrow$

$-\frac{1}{x} = t - 1 \Leftrightarrow x = \frac{1}{1-t}$

$\dot{z} = \frac{2z}{1-t} + \frac{t}{(1-t)^3}$

I. $\dot{z} = \frac{2z}{1-t}$

$\int \frac{dz}{z} = 2\int \frac{dt}{1-t} \Leftrightarrow z = \frac{C}{(1-t)^2}$

II. $C = C(t)$

$\frac{\dot{C}(1-t)^2 + \cancel{2(1-t)C}}{(1-t)^4} = \frac{2\cancel{C}}{(1-t)^3} + \frac{t}{(1-t)^3} \Leftrightarrow$

$\Leftrightarrow \frac{\dot{C}}{(1-t)^2} = \frac{t}{(1-t)^3} \Leftrightarrow C(t) = \int \frac{t-1+1}{(1-t)} dt =$

$= -\int dt + \int \frac{dt}{(1-t)} =$

$= -t - \ln|1-t| + C_1$

$z(x) = \frac{C_1 - t - \ln|1-t|}{(1-t)^2}$

$z(0) = \frac{C_1}{1} = 1 \Rightarrow z(x) = \frac{1 - t - \ln|1-t|}{(1-t)^2}$

Ответ: $\left.\frac{\partial x}{\partial \mu}\right|_{\mu=0} = \frac{1 - t - \ln|1-t|}{(1-t)^2}$

№1072

$\begin{cases} \ddot{x} - \dot{x} = (x+1)^2 - \mu x^2 \\ x(0) = \frac{1}{2} \\ \dot{x}(0) = -1 \end{cases}$ $\left.\frac{\partial x}{\partial \mu}\right|_{\mu=1} - ?$

1) Пусть $z = \left.\frac{\partial x}{\partial \mu}\right|_{\mu=1}$

$\begin{cases} \ddot{z} - \dot{z} = 2(x+1)z - x^2 - 2\mu x z \\ z(0) = 0 \\ \dot{z}(0) = 0 \end{cases}$

2) $\mu = 1$

$$\begin{cases} x(0) = 1/2 \\ \dot{x}(0) = -1 \end{cases} \qquad \begin{cases} z(0) = 0 \\ \dot{z}(0) = 0 \end{cases}$$

$$\ddot{x} - \dot{x} = 2x + 1$$

$$\lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda = 2; -1$$

$$x_{oo}(t) = C_1 e^{2t} + C_2 e^{-t}$$

$$x(t) = C_1 e^{2t} + C_2 e^{-t} - \frac{1}{2}$$

$$\begin{cases} x(0) = 1/2 \\ \dot{x}(0) = -1 \end{cases} \Leftrightarrow \begin{cases} C_1 + C_2 - 1/2 = 1/2 \\ 2C_1 - C_2 = -1 \end{cases} \Leftrightarrow \begin{cases} C_1 + C_2 = 1 \\ 3C_1 = 0 \end{cases} \Leftrightarrow \begin{cases} C_1 = 0 \\ C_2 = 1 \end{cases}$$

$$x(t) = e^{-t} - \frac{1}{2}$$

$$\ddot{z} - \dot{z} = 2z - \left(e^{-t} - \tfrac{1}{2}\right)^2$$

$$\lambda^2 - \lambda - 2 = 0$$

$$z_{oo}(t) = C_1 e^{2t} + C_2 e^{-t}$$

$$C_1 = C_1(t), \quad C_2 = C_2(t)$$

$$\begin{cases} \dot{C}_1 e^{2t} + \dot{C}_2 e^{-t} = 0 \\ 2\dot{C}_1 e^{2t} - \dot{C}_2 e^{-t} = -(e^{-t} - 1/2)^2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \dot{C}_1 e^{3t} + \dot{C}_2 = 0 \\ 2\dot{C}_1 e^{3t} - \dot{C}_2 = \left(-e^{-2t} - \frac{1}{4} + e^{-t}\right) e^{t} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \dot{C}_2 = -\dot{C}_1 e^{3t} \\ 3\dot{C}_1 e^{3t} = 1 - \frac{1}{4} e^{t} - e^{-t} \end{cases} \Leftrightarrow \begin{cases} \dot{C}_2 = -\dot{C}_1 e^{3t} \\ \dot{C}_1 = \frac{1}{3}\left(e^{-3t} - \frac{1}{4} e^{-2t} - e^{-4t}\right) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \dot{C}_2 = -\frac{1}{3}\left(1 - \frac{1}{4} e^{t} - e^{-t}\right) \\ C_1 = \frac{1}{3}\left(-\frac{1}{3} e^{-3t} + \frac{1}{8} e^{-2t} + \frac{1}{4} e^{-4t}\right) + C \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} C_1 = -\frac{1}{9} e^{-3t} + \frac{1}{24} e^{-2t} + \frac{1}{12} e^{-4t} + C_3 \cdot \left(-\frac{1}{72}\right) \\ C_2 = -\frac{1}{3} t + \frac{1}{12} e^{t} - \frac{1}{3} e^{-t} + C_4 \cdot \left(-\frac{1}{72}\right) \end{cases}$$

$$z(t) = -\frac{1}{72}\left(8e^{-t} - 3 - 6e^{-2t} + 24te^{-t} - 6 + 24e^{-2t} + C_3 e^{2t} + C_4 e^{-t}\right)$$

$$= -\frac{1}{72}\left(e^{-t}(8 + 24t) - 9 + e^{-2t}(-6 + 24)\right) + C_3 e^{2t} + C_4 e^{-t}$$

$$\begin{cases} 8 - 3 - 6 - 6 + 24 + C_3 + C_4 = 0 \\ 8 + 12 + 24 - 48 + 2C_3 - C_4 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} C_3 + C_4 = -17 \\ 2C_3 - C_4 = 20 \end{cases} \Leftrightarrow \begin{cases} 3C_3 = 3 \\ C_4 = 2C_3 - 20 \end{cases} \Leftrightarrow \begin{cases} C_3 = 1 \\ C_4 = -18 \end{cases}$$

$$z(t) = -\frac{1}{72}\left(e^{-t}(24t - 10) - 9 + 18e^{-2t} + e^{2t}\right) =$$

$$= \frac{1}{8} + e^{-t}\left(-\frac{t}{3} + \frac{5}{36}\right) - \frac{1}{4} e^{-2t} - \frac{1}{72} e^{2t}$$

Ответ: $\left.\frac{\partial x}{\partial \mu}\right|_{\mu=0} = \frac{1}{8} + e^{-t}\left(\frac{5}{36} - \frac{t}{3}\right) - \frac{1}{4} e^{-2t} - \frac{1}{72} e^{2t}$

№ 1071

$$\begin{cases} \dot{x} = x + y \\ \dot{y} = 2x + \mu y^2 \\ x(0) = 1 + \mu \\ y(0) = -2 \end{cases} \qquad \left.\frac{\partial y}{\partial \mu}\right|_{\mu=0} - ?$$

1) $\frac{\partial x}{\partial \mu} = u \qquad \frac{\partial y}{\partial \mu} = v$

$$\begin{cases} \dot v = 2u + y^2 + 2\mu y v \\ u(0) = 1 \\ v(0) = 0 \end{cases}$$

2) $\mu = 0$

$$\begin{cases} \dot x = x + y \\ \dot y = 2x \\ x(0) = 1 \\ y(0) = -2 \end{cases} \qquad \begin{cases} \dot u = u + v \\ \dot v = 2u + y^2 \\ u(0) = 1 \\ v(0) = 0 \end{cases}$$

$x = \frac{1}{2}\dot y \Rightarrow \quad \dot x = x + y \Leftrightarrow \frac{1}{2}\ddot y = \frac{1}{2}\dot y + y \Leftrightarrow$

$\Leftrightarrow \ddot y - \dot y - 2y = 0$

$\lambda^2 - \lambda - 2 = 0 \quad \Leftrightarrow \lambda = 2; -1$

$y = c_1 e^{2t} + c_2 e^{-t}$

$$\begin{cases} c_1 + c_2 = -2 \\ 2c_1 - c_2 = 2 \end{cases} \Leftrightarrow \begin{cases} 3c_1 = 0 \\ c_2 = -2 \end{cases}$$

$y = -2e^{-t}$

$$\begin{cases} \dot u = u + v \\ \dot v = 2u + 4e^{-2t} \\ u(0) = 1 \\ v(0) = 0 \end{cases} \Rightarrow u = \frac{1}{2}(\dot v - 4e^{-2t})$$

$\frac{1}{2}(\ddot v + 8e^{-2t}) = \frac{1}{2}(\dot v - 4e^{-2t}) + v \quad \Leftrightarrow$

$\Leftrightarrow \ddot v + 8e^{-2t} = \dot v + 2v - 4e^{-2t} \Leftrightarrow$

$\Leftrightarrow \ddot v - \dot v - 2v = -12e^{-2t}$

I. $\lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda = 2; -1$

$v_{oo} = c_1 e^{2t} + c_2 e^{-t}$

$$\begin{cases} \dot c_1 e^{2t} + \dot c_2 e^{-t} = 0 \\ 2\dot c_1 e^{2t} - \dot c_2 e^{-t} = -12e^{-2t} \end{cases} \Leftrightarrow \begin{cases} \dot c_1 e^{3t} + \dot c_2 = 0 \\ 2\dot c_1 e^{3t} - \dot c_2 = -12e^{-t} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \dot c_1 e^{3t} = -4e^{-t} \\ \dot c_2 = -\dot c_1 e^{3t} \end{cases} \Leftrightarrow \begin{cases} \dot c_1 = -4e^{-4t} \\ \dot c_2 = 4e^{-t} \end{cases} \Leftrightarrow \begin{cases} c_1 = e^{-4t} + c_3 \\ c_2 = -4e^{-t} + c_4 \end{cases}$$

$v_H = e^{-2t} + c_3 e^{2t} - 4e^{-2t} + c_4 e^{-t}$

$$\begin{cases} 1 + c_3 - 4 + c_4 = 0 \\ \frac{1}{2}(-4 + 2 + 2c_3 + 8 - c_4) = 1 \end{cases} \Leftrightarrow \begin{cases} c_3 + c_4 = 3 \\ 2c_3 - c_4 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} c_3 = 1 \\ c_4 = 2 \end{cases}$$

$v = e^{2t} - 3e^{-2t} + 2e^{-t}$

Ответ: $\left.\frac{\partial y}{\partial \mu}\right|_{\mu=0} = e^{2t} - 3e^{-2t} + 2e^{-t}$

№1075

$$\begin{cases} y' = \frac{2}{y} - 5\mu x \\ y(1) = 2 \end{cases} \qquad y(x) = a_0(x) + a_1(x)\mu + a_2(x)\mu^2 + \dots$$

$$\begin{cases} a_0' + a_1'\mu + a_2'\mu^2 + \dots = 2(a_0 + a_1\mu + a_2\mu^2 + \dots)^{-1} - 5\mu x \\ a_0(1) + a_1(1)\mu + a_2(1)\mu^2 + \dots = 2 \end{cases}$$

$$2(a_0 + a_1\mu + a_2\mu^2 + \dots)^{-1} = \frac{2}{a_0} - \left.\frac{2(a_1 + 2a_2\mu + 3a_3\mu^2 + \dots)}{(a_0 + a_1\mu + a_2\mu^2 + \dots)^2}\right|_{\mu=0} \cdot \mu -$$

$$- 2\cdot\frac{(2a_2 + 6a_3\mu + \dots)(a_0 + a_1\mu + a_2\mu^2 + \dots)^2 - 2(a_0 + a_1\mu + a_2\mu^2 + \dots)(a_1 + 2a_2\mu + 3a_3\mu^2 + \dots)^2}{(a_0 + a_1\mu + a_2\mu^2 + \dots)^4}$$

$$= \frac{2}{a_0} - \frac{2a_1}{a_0^2}\mu - \frac{2a_2 a_0^2 - 2a_0 a_1^2}{\dots}\mu^2 = \frac{2}{a_0} - \frac{2a_1}{a_0^2}\mu - \frac{2a_0 a_2 - 2a_1^2}{a_0^3}\mu^2 \dots$$

$\mu^0: \begin{cases} a_0' = \frac{2}{a_0} \\ a_0(1) = 2 \end{cases} \Rightarrow \int a_0 da_0 = 2\int dx \Rightarrow a_0^2 = 4x + C$

$a_0^2(1) = 4 = 4 + C \Rightarrow C = 0 \Rightarrow \boxed{a_0 = 2\sqrt{x}}$

$\mu^1: \begin{cases} a_1' = -\frac{2a_1}{a_0^2} - 5x \\ a_1(1) = 0 \end{cases}$

$\mu^2: \begin{cases} a_2' = \frac{2a_1^2 - 2a_2 a_0}{a_0^3} \\ a_2(1) = 0 \end{cases}$

$a_1' = -\frac{2a_1}{4x} - 5x$

I. $a_1' = -\frac{a_1}{2x} \Leftrightarrow \int \frac{da_1}{a_1} = -\frac{1}{2}\int \frac{dx}{x} \Leftrightarrow a_1 = Cx^{-1/2}$

II. $C = C(t)$

$C' x^{-1/2} - \cancel{\frac{1}{2} C x^{-3/2}} = \cancel{-\frac{1}{2} C x^{-3/2}} - 5x \Leftrightarrow C' = -5x^{3/2} \Leftrightarrow$

$\Leftrightarrow C = -5 \cdot \frac{2}{5} x^{5/2} + C_1 = -2x^{5/2} + C_1$

$a_1 = -2x^2 + C_1 x^{-1/2}$

$a_1(1) = -2 + C_1 = 0 \Rightarrow C_1 = 2$

$a_1(x) = \boxed{-2x^2 + 2x^{-1/2}}$

$a_2' = \frac{2 \cdot 4(\sqrt{x}^{-1} - x^2)^2}{8x\sqrt{x}} - 2\frac{a_2 \cdot 2\sqrt{x}}{8x\sqrt{x}} \Leftrightarrow$

$\Leftrightarrow a_2' = \frac{\frac{1}{x} + x^4 - 2x^{3/2}}{\sqrt{x}\, x} - \frac{a_2}{2x}$

I. $\int \frac{da_2}{a_2} = -\frac{1}{2}\int \frac{dx}{x} \Leftrightarrow a_2 = Cx^{-1/2}$

II. $C = C(t)$

$C' x^{-1/2} = -x + \frac{1}{2} x^{-1/2} + \frac{1}{2} x^{5/2} \Leftrightarrow C' = -x^{3/2} + \frac{1}{2} + \frac{1}{2}x^3$

$C = -\frac{2}{5} x^{5/2} + \frac{1}{2}x + \frac{1}{4}x^4$

$C' x^{-1/2} = \frac{1}{2}(x^{-2} + x^{3} - 2) \Leftrightarrow$

$\Leftrightarrow C' = (x^{-2} + x^3 - 2x^{1/2}) \Leftrightarrow$

$\Leftrightarrow C = -x^{-1} + \frac{1}{4}x^4 - \frac{4}{3}x^{3/2} + C_1$

$a_2 = C_1 x^{-1/2} - x^{-3/2} + \frac{1}{4}x^{7/2} - \frac{4}{3}x$

$a_2(1) = C_1 - 1 + \frac{1}{4} - \frac{4}{3} = 0 \Leftrightarrow C_1 = \frac{25}{12}$

Ответ: $y(x) = 2\sqrt{x} + 2\mu(x^{-1/2} - x^2) + \mu^2\left(\frac{x^{7/2}}{4} - \frac{4}{3}x + \frac{25}{12}x^{-1/2} - x^{-3/2}\right)$

№ 1078

$\begin{cases} y' = e^{y-x} + \mu y \\ y(0) = -\mu \end{cases} \qquad y(x) = a_0(x) + a_1(x)\mu + a_2(x)\mu^2 + \dots$

$\begin{cases} a_0' + a_1'\mu + a_2'\mu^2 + \dots = e^{a_0 + a_1\mu + a_2\mu^2 + \dots - x} + \mu(a_0 + a_1\mu + a_2\mu^2 + \dots) \\ a_0(0) + a_1(0)\mu + a_2(0)\mu^2 + \dots = -\mu \end{cases}$

$e^{a_0 + a_1\mu + a_2\mu^2 + \dots - x} = e^{a_0 - x} + (a_1 + 2a_2\mu + 3a_3\mu^2 + \dots) e^{a_0 + a_1\mu + a_2\mu^2 \dots - x}\Big|_{\mu=0}\mu$

$+ \frac{2a_2 e^{a_0 - x} + a_1^2 e^{a_0 - x} + \dots}{2}\mu^2 = e^{a_0 - x} + a_1 e^{a_0 - x}\mu + (a_2 e^{a_0 - x} + \frac{1}{2}a_1^2 e^{a_0 - x})\mu^2 + \dots$

$\mu^0: \begin{cases} a_0' = e^{a_0 - x} \\ a_0(0) = 0 \end{cases} \Rightarrow \int \frac{da_0}{e^{a_0}} = \int e^{-x} dx \Leftrightarrow e^{-a_0} = e^{-x} + C$

$1 = 1 + C \Rightarrow C = 0 \Rightarrow \boxed{a_0 = x}$

$\mu^1: \begin{cases} a_1' = a_1 e^{a_0 - x} + a_0 \\ a_1(0) = -1 \end{cases}$

$\mu^2: \begin{cases} a_2' = e^{a_0 - x}(a_2 + \frac{1}{2}a_1^2) + a_1 \\ a_2(0) = 0 \end{cases}$

$a_1' = a_1 + x$

I. $a_1' = a_1 \Leftrightarrow \int \frac{da_1}{a_1} = \int dx \Leftrightarrow a_1 = Ce^x$

II. $C = C(t) \qquad C' e^x = x \Leftrightarrow C = \int \frac{x}{e^x} dx = -xe^{-x} + \int e^{-x} dx =$

$$a_1(0) = C_1 - 1 = -1 \Rightarrow C_1 = 0 \Rightarrow \boxed{a_1 = -x-1}$$

$$a_2' = a_2 - x - 1 + \frac{1}{2}(x+1)^2 = a_2 \cancel{-x} - 1 + \frac{x^2}{2} + \frac{1}{2} \cancel{+x} = a_2 + \frac{x^2}{2} - \frac{1}{2}$$

I. $a_2' = a_2 \Rightarrow a_2 = Ce^x$

II. $C'e^x = \frac{x^2}{2} - \frac{1}{2} \Leftrightarrow C' = \frac{1}{2}\,\frac{x^2-1}{e^x} \Leftrightarrow C = -\frac{1}{2}\int (x^2-1)\,de^{-x} =$

$$= -\frac{1}{2}(x^2-1)e^{-x} + \frac{1}{2}\int e^{-x}\cdot 2x\,dx = -\frac{1}{2}(x^2-1)e^{-x} - e^{-x}(x+1) + C_1$$

$$a_2(x) = C_1e^x - x - 1 - \frac{x^2}{2} + \frac{1}{2}$$

$$a_2(0) = C_1 - \frac{1}{2} = 0 \Rightarrow C_1 = \frac{1}{2}$$

$$a_2(x) = \frac{e^x}{2} - x - \frac{1}{2} - \frac{x^2}{2}$$

Ответ: $y(x) = x - \mu(x+1) + \frac{\mu^2}{2}\left(e^x - 1 - x^2 - 2x\right)$

$$\begin{cases} y' = f(x,y) \\ y(0) = y_0 \end{cases}$$

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$$

$a_i = const$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

№1091

$$\begin{cases} y' = y^2 - x \\ y(0) = 1 \end{cases} \qquad y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots$$

$$\begin{cases} a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \dots = \\ = (a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots)^2 - x \\ a_0 = 1 \end{cases}$$

$$\begin{array}{l|l} x^0: & a_1 = a_0^2 \\ x^1: & 2a_2 = 2a_0a_1 - 1 \\ x^2: & 3a_3 = a_1^2 + 2a_0a_2 \\ x^3: & 4a_4 = 2a_1a_2 + 2a_0a_3 \end{array} \Leftrightarrow \begin{cases} a_0 = 1 \\ a_1 = 1 \\ a_2 = 1/2 \\ a_3 = 2/3 \\ a_4 = 7/12 \end{cases}$$

Ответ: $y(x) = 1 + x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{7x^4}{12} + \dots$

№1092

$$\begin{cases} y' = x + \frac{1}{y} \\ y(0) = 1 \end{cases} \qquad y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$$

$$\begin{cases} (a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \dots)(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots) = \\ = x(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots) + 1 \\ a_0 = 1 \end{cases}$$

$$\begin{array}{l|l} x^0: & a_0a_1 = 1 \\ x^1: & a_1^2 + 2a_2a_0 = a_0 \\ x^2: & 3a_3a_0 + a_1a_2 = a_1 \\ x^3: & 4a_4a_0 + a_1a_3 = a_2 \end{array} \Leftrightarrow \begin{cases} a_0 = 1 \\ a_1 = 1,\ a_2 = 0 \\ 3a_3 + a_2 = 1 \\ 4a_4 + a_3 = a_2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} a_1 = a_0 = 1,\ a_2 = 1/2 \\ 4a_4 + 4a_3 = 1 \\ a_3 = 1/3 \end{cases} \Leftrightarrow \begin{cases} a_0 = 1 \\ a_1 = 1 \\ a_2 = 0 \\ a_3 = 1/3 \\ a_4 = -1/3 \end{cases}$$

Ответ: $y(x) = 1 + x + \frac{x^3}{3} - \frac{x^4}{3} + \dots$

№1102

$$(1-x^2)y'' - 4xy' - 2y = 0 \qquad y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\cancel{(1-x^2)\sum n a_n x^n}$$

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 4x\sum_{n=1}^{\infty} n a_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 4\sum_{n=1}^{\infty} n a_n x^n - 2\sum_{n=0}^{\infty} a_n x^n = 0 \Leftrightarrow$$

$$\Leftrightarrow 2a_2 + 6a_3 x + \sum_{n=4}^{\infty} n(n-1)a_n x^{n-2} - 4a_1 x - 4\sum_{n=2}^{\infty} n a_n x^n - 2a_0 - 2a_1 x - 2\sum_{n=2}^{\infty} a_n x^n = 0 \Leftrightarrow$$

$$+\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$\Leftrightarrow 2(a_2-a_0) + x(6a_3 - 4a_1 - 2a_1) + \sum_{n=0}^{\infty}\left((n+4)(n+3)a_{n+4} - 4(n+2)a_{n+2} - 2a_{n+2}\right)x^{n+2} = 0 \Leftrightarrow$$

$$\Leftrightarrow 2(a_2-a_0) + (6a_3 - 6a_1) + \sum_{n=2}^{\infty} x^n\left(\underbrace{(n+2)(n+1)a_{n+2}}_{-n(n-1)a_n} - 4na_n - 2a_n\right) = 0$$

x^0: $a_2 - a_0 = 0$

x^1: $a_3 - a_1 = 0$

x^n: $a_{n+2} = a_n$

1) $a_0 = 0$, $a_1 = 1$

$$y(x) = x + x^3 + x^5 + \dots = \frac{x}{1-x^2}$$

2) $a_0 = 1$, $a_1 = 0$

$$y(x) = 1 + x^2 + x^4 + x^6 + \dots = \frac{1}{1-x^2}$$

Ответ: $y(x) = C_1 \frac{x}{1-x^2} + C_2 \frac{1}{1-x^2}$

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad r\in\mathbb{R}, \quad a_0 \neq 0$$

№1110

$$xy'' + 2y' + xy = 0 \qquad y = \sum_{n=0}^{\infty} a_n x^{n+r} = a_0x^r + a_1x^{r+1} + a_2x^{2+r} + a_3x^{3+r} + \dots$$

$$x(a_0x^r + a_1x^{1+r} + a_2x^{2+r} + a_3x^{3+r} + \dots) + 2(ra_0x^{r-1} + (r+1)a_1x^r + (r+2)a_2x^{1+r} + a_3(r+3)x^{2+r} + \dots) +$$

$$+ x(r(r-1)a_0x^{r-2} + r(r+1)a_1x^{r-1} + (r+1)(r+2)a_2x^r + (r+2)(r+3)a_3x^{r+1} + \dots) = 0$$

$$\begin{cases} x^{r-1}: & 2ra_0 + r(r-1)a_0 = 0 \\ x^r: & 2(r+1)a_1 + r(r+1)a_1 = 0 \\ x^{r+1}: & a_0 + 2(r+2)a_2 + (r+1)(r+2)a_2 = 0 \\ \dots & \\ x^{r+n}: & a_{n-1} + 2(r+n+1)a_{n+1} + (r+n)(r+n+1)a_{n+1} = 0 \end{cases}$$

$a_0(r^2+r) = 0$

$a_1(r^2+3r+2) = 0$

$$a_{n-1} + a_{n+1}(r+1+n)(2+r+n) = 0 \Rightarrow a_{n+1} = -\frac{a_{n-1}}{(r+n+1)(r+n+2)}$$

1) $r = 0$

$a_1 = 0$

$$a_{n+1} = \frac{-a_{n-1}}{(n+1)(n+2)}$$

$$a_0 = 1, \quad a_2 = \frac{-1}{6}, \quad a_4 = \frac{\frac{1}{6}}{5\cdot 4} = \frac{1}{120}, \quad a_n = \frac{(-1)^n}{(2n+1)!}$$

$$y(x) = \sum \frac{(-1)^n}{(2n+1)!} x^{2n} = \frac{\sin x}{x}$$

$\Phi[y] = \int_{x_0}^{x_1} F(x,y,y')dx, \quad y(x_0) = y_0, \; y(x_1) = y_1$

2) $\nu = -1$

$a_1 \cdot 0 = 0$, $a_1 = 0$
$a_0 = 1$

$$a_{n+1} = \frac{-a_{n-1}}{n(n+2)}$$

$$a_{2n} = \frac{(-1)^n}{(2n)!} \qquad y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n-1} = \frac{\cos x}{x}$$

Д.з.

№ ~~1078~~ 1077, 78, 93, 94, 1104
1113

№ 1077

$$\begin{cases} y' = \frac{6\mu}{x} - y^2 \\ y(1) = 1 + 3\mu \end{cases} \qquad y(x) = a_0(x) + a_1(x)\mu + a_2(x)\mu^2 + \ldots$$

$$\begin{cases} a_0' + a_1'\mu + a_2'\mu^2 + \ldots = \frac{6\mu}{x} - (a_0 + a_1\mu + a_2\mu^2 + \ldots)^2 \\ a_0(1) + a_1(1)\mu + a_2(1)\mu^2 + \ldots = 1 + 3\mu \end{cases}$$

μ^0: $\begin{cases} a_0' = -a_0^2 \\ a_0(1) = 1 \end{cases}$

μ^1: $\begin{cases} a_1' = \frac{6}{x} - 2a_0 a_1 \\ a_1(1) = 3 \end{cases}$

μ^2: $\begin{cases} a_2' = -a_1^2 - 2a_0 a_2 \\ a_2(1) = 0 \end{cases}$

1) $\int \frac{da_0}{a_0^2} = -\int dx \iff -\frac{1}{a_0} = -x + C \Rightarrow -1 = -1 + C \Rightarrow C = 0$

$$\boxed{a_0(x) = \frac{1}{x}}$$

2) $a_1' = \frac{6}{x} - \frac{\ldots}{x}$, $a_1' = -\frac{2a_1}{x}$

$$\int \frac{da_1}{a_1} = -2\int \frac{dx}{x} \iff a_{1\,од.} = Cx^{-2}$$

$$a_1 = 3 + Cx^{-2} \Rightarrow 3 = 3 + Cx^{-2} \Rightarrow C = 0$$

$$\Rightarrow \boxed{a_1 = 3}$$

3) $a_2' = -9 - \frac{2a_2}{x}$

I. $\int \frac{da_2}{a_2} = -2\int \frac{dx}{x} \iff a_{2\,од.} = Cx^{-2}$

II. $C = C(t)$

$$C'x^{-2} - \cancel{2Cx^{-3}} = -9 - \cancel{2Cx^{-3}} \iff C' = -9x^2$$

$$\iff C = -3x^3 + C_1 \Rightarrow a_2 = -3x + C_1 x^{-2}$$

$$0 = -3 + C_1 \Rightarrow C_1 = 3 \Rightarrow \boxed{a_2 = -3x + \frac{3}{x^2}}$$

Ответ: $y(x) = \frac{1}{x} + 3\mu + \mu^2\left(\frac{3}{x^2} - 3x\right) + \ldots$

№ 1093

$$\begin{cases} y' = y + xe^y \\ y(0) = 0 \end{cases} \qquad y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

$$\begin{cases} a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \ldots \\ \ldots + x\left(1 + (a_0 + a_1 x + \ldots) + \frac{1}{2}(a_0 + a_1 x + \ldots)^2 + \ldots\right. \\ a_0 = 0 \qquad \ldots + \end{cases}$$

x^0: $a_1 = a_0 = 0$ $\qquad x^2$: $3a_3 = a_2 \Rightarrow a_3 = \frac{1}{6}$

x^1: $2a_2 \ldots \frac{1}{\ldots}$ $\quad x^3$: $4 \ldots \frac{1}{\ldots}$

Ответ: $y(x) = \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \dots$

№ 1094

$$\begin{cases} y' = 2x + \cos y \\ y(0) = 0 \end{cases} \qquad y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$$

$$\begin{cases} a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots = 2x + 1 - \frac{1}{2}(a_0 + a_1x + a_2x^2 + \dots)^2 + \frac{1}{24}(a_0 + a_1x + a_2x^2 \dots)^4 \\ a_0 = 0 \end{cases}$$

$x^0: a_1 = 1 - \frac{a_0^2}{2} + \frac{a_0^4}{24} = 1$

$x^1: 2a_2 = 2 - \frac{1}{2}\cdot 2a_0a_1 = 2 \Rightarrow a_2 = 1$

$x^2: 3a_3 = -\frac{1}{2}a_1^2 - a_0a_2 \Rightarrow a_3 = -\frac{1}{6}$

$x^3: 4a_4 = -a_1a_2 - \cancel{\frac{1}{2}a_2^2} \Rightarrow a_4 = -\frac{1}{4}$

Ответ: $y(x) = x + x^2 - \frac{x^3}{6} - \frac{x^4}{4} + \dots$

№ 1104

$(1-x)y'' - 2y' + y = 0 \qquad y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$(1-x)\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} - 2\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_nx^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} - \sum_{n=2}^{\infty}(n^2 - n + 2n)a_nx^{n-1} - 2a_1 + \sum_{n=0}^{\infty} a_nx^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}(n+2)(n+3)a_{n+1}x^n - 2a_1 + \sum_{n=0}^{\infty}a_nx^n = 0$$

$x^0: 2a_2 - 2a_1 + a_0 = 0$

$x^n: (n+2)(n+1)(a_{n+2} - a_{n+1}) + a_n = 0 \Rightarrow a_{n+2} = a_{n+1} - \frac{a_n}{(n+2)(n+1)}$

1) $a_0 = 0, \ a_1 = 1$

$a_2 = 1 \quad a_3 = \frac{5}{6} \quad a_4 = \frac{3}{4}$

2) $a_0 = 1, \ a_1 = 0$

$a_2 = -\frac{1}{2}, \ a_3 = -\frac{1}{2}, \ a_4 = -\frac{11}{24}$

Ответ: 1) $y(x) = x + x^2 + \frac{5x^3}{6} + \frac{3x^4}{4} + \dots$

2) $y(x) = 1 - \frac{x^2}{2} - \frac{x^3}{3} - \frac{11x^4}{24} + \dots$

№ 1113

$x^2y'' - x^2y' + (x-2)y = 0 \qquad y(x) = \sum_{n=0}^{\infty} a_nx^{n+r}$

$$(x-2)\sum_{n=0}^{\infty} a_nx^{n+r} - x^2\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1} + x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2} = 0$$

$$\sum_{n=0}^{\infty} a_nx^{n+r+1} - 2\sum_{n=0}^{\infty} a_nx^{n+r} - \sum_{n=0}^{\infty}(n+r)a_nx^{n+r+1} + \sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(1-n-r)a_nx^{n+r+1} + \sum_{n=0}^{\infty}\big((n+r)(n+r-1) - 2\big)a_nx^{n+r} = 0$$

$x^r: (r^2 - r - 2)a_0 = 0 \Leftrightarrow r = 2; -1$

$x^{r+1}: (1-r)a_0 + (r^2 + r - 2)a_1 = 0$

$x^{n+r}: (2-n-r)a_{n-1} + \big((n+r)(n+r-1) - 2\big)a_n = 0 \Rightarrow a_n = \frac{(n+r-2)a_{n-1}}{(n+r)(n+r-1) - 2}$

1) $r = -1, \ a_0 = 1$

$-2 - 2a_1 = 0 \Leftrightarrow a_1 = 1, \ a_n = \frac{(n-3)a_{n-1}}{(n-1)(n-2) - 2} = \frac{(n-3)a_{n-1}}{n^2 - 3n} = \frac{a_{n-1}}{n}$

$a_2 = \frac{1}{2}$

2) $n=2,\ a_0=1$ $\quad a_n = \frac{}{(n+2)(n+1)-2} = \frac{}{n^2+3n} =$

$$= \frac{a_{n-1}}{n+3}$$

$$a_1 = \frac{1}{4},\ a_2 = \frac{1}{4\cdot 5},\ a_3 = \frac{1}{4\cdot5\cdot6},$$

$$a_n = \frac{3!}{(n+3)!}$$

$$y(x) = \sum_{n=0}^{\infty} \frac{3!}{(n+3)!} x^{n+2}$$

Ответ: 1) $y = \frac{1}{x} + 1 + \frac{x}{2}$

2) $y = \sum_{n=0}^{\infty} \frac{3!\, x^{n+2}}{(n+3)!}$

12.04.17

№ 1141

$$\begin{cases} y' = \frac{x}{z} \\ z' = -\frac{x}{y} \end{cases}$$

$$\begin{cases} \frac{dy}{dz} = -\frac{y}{z} \\ \frac{dz}{dx} = -\frac{x}{y} \end{cases} \Leftrightarrow \begin{cases} \int\frac{dy}{y} = -\int\frac{dz}{z} \\ \frac{dz}{dx} = -\frac{x}{y} \end{cases} \Leftrightarrow \begin{cases} y = \frac{c}{z} \\ \frac{dz}{dx} = -\frac{zx}{c} \end{cases}$$

$$\int\frac{dz}{z} = -\frac{1}{c}\int dx\, x \Rightarrow z = c_1 e^{-\frac{x^2}{2c}}$$

$$y = \frac{c\, e^{\frac{x^2}{2c}}}{c_1}$$

Ответ: $\begin{cases} z = c_1 e^{-\frac{x^2}{2c}} \\ y = \frac{c\, e^{\frac{x^2}{2c}}}{c_1} \end{cases}$

№ 1142

$$\begin{cases} y' = \frac{y^2}{z-x} \\ z' = y+1 \end{cases} \Leftrightarrow \begin{cases} z = x + \frac{y^2}{y'} \\ 1 + \frac{-y''\cdot y^2 + 2y(y')^2}{(y')^2} = y+1 \end{cases} \Leftrightarrow$$

1) $\begin{cases} y=0 \text{ — реш.} \\ z = x + C \end{cases}$

$$\Leftrightarrow \begin{cases} z = x + \frac{y^2}{y'} \\ y\left(2(y')^2 - y\cdot y'' - (y')^2\right) = 0 \end{cases}$$

$$t(y) = y',\quad y'' = t'\cdot t$$

$$t^2 - y\cdot t' t = 0$$

1) $t = 0$

2) $t - yt' = 0 \Leftrightarrow \int\frac{dt}{t} = \int\frac{dy}{y} \Leftrightarrow t = Cy$

$$y' = Cy \Leftrightarrow \int\frac{dy}{y} = C\int dx \Leftrightarrow y = C_1 e^{Cx}$$

$$z = x + \frac{C_1^2 e^{2Cx}}{C\cdot C_1 e^{Cx}} = x + \frac{C_1}{C} e^{Cx}$$

Ответ: $\begin{cases} y = C_1 e^{Cx} \\ z = x + \frac{C_1 e^{Cx}}{C} \end{cases}$

$$\Phi[y] = \int^{x_1} F(x,y,y')dx,\quad y(x_0)=y_0,\ y(x_1)=y_1$$

$\begin{cases} y' = \frac{x}{z} \\ z' = -\frac{x}{y} \end{cases} \Rightarrow \begin{cases} \frac{dy}{dx} = \frac{x}{z} \\ \frac{dz}{dx} = -\frac{x}{y} \end{cases} \Rightarrow \quad dx = \frac{z\,dy}{x}, \quad dx = -\frac{y\,dz}{x}$

$$\boxed{\frac{dx}{1} = \frac{dy}{x/z} = \frac{dz}{-x/y}} = dt$$

$\bar{x} = (x_1(t), x_2(t), \dots, x_n(t))$

$\dot{x} = f(x)$ $\quad \underline{\Phi_n(x) = C_n}$ – первый интеграл системы

№ 1161

$$\begin{cases} \frac{dx}{dt} = \frac{x^2 - t}{y} \\ \frac{dy}{dt} = -x \end{cases} \qquad \varphi_1 = t^2 + 2xy, \quad \varphi_2 = x^2 - ty$$

№ 1163

$$\frac{dx}{y} = -\frac{dy}{x} = \frac{dz}{u} = -\frac{du}{z} = dt \qquad \varphi = yz - ux$$

$\cancel{dx}\ \dot{x} = y, \quad \dot{y} = -x, \quad \dot{u} = -z, \quad \dot{z} = u$

$$\dot{\varphi} = \dot{y}z + y\dot{z} - \dot{u}x - u\dot{x} = -\cancel{xz} + uy + \cancel{zx} - uy = 0$$

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \dots = \frac{a_n}{b_n} = t \Leftrightarrow \{a_i = t b_i\} =$$

$$= \frac{k_1 a_1 + k_2 a_2 + \dots + k_n a_n}{k_1 b_1 + k_2 b_2 + \dots + k_n b_n} = \frac{k_1 t b_1 + \dots + k_n t b_n}{k_1 b_1 + \dots + k_n b_n} = t$$

№ 1147

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{z}$$

1) $\frac{dx}{y} = \frac{dy}{x} \Leftrightarrow \int x\,dx = \int y\,dy \Leftrightarrow x^2 = y^2 + C_1 \Rightarrow \boxed{C_1 = x^2 - y^2}$

2) ① + ② = ③

$$\frac{dx + dy}{y + x} = \frac{dz}{z} = \frac{d(x+y)}{x+y} \Rightarrow x + y = C_2 z \Rightarrow \boxed{C_2 = \frac{x+y}{z}}$$

№ 1148

$$\frac{dx}{y+z} = \frac{dy}{x+z} = \frac{dz}{x+y}$$

$$\frac{dx - dy}{y - x} = \frac{dy - dz}{z - y}$$

$$\int \frac{d(x-y)}{x-y} = \int \frac{d(y-z)}{y-z} \Rightarrow \boxed{\frac{x-y}{y-z} = C_1}$$

$$\frac{dx + dy + dz}{2(x+y+z)} = \frac{d(x-y)}{-(x-y)} \Rightarrow \boxed{(x-y)(x+y+z)^{1/2} = C_2}$$

$\Phi[y] = \int_{x_0}^{x_1} F(x, y, y')\,dx, \quad y(x_0) = y_0, \ y(x_1) = y_1$

$$\Phi_n(x_1(t), \dots, x_n(t)) = C_n$$

$$\text{rang} \left\| \frac{\partial \Phi_i}{\partial x_j} \right\|_{\substack{i=\overline{1,n} \\ j=\overline{1,n-1}}} = n-1$$

№1149

$$\frac{dx}{y-x} = \frac{dy}{x+y+z} = \frac{dz}{x-y}$$

$$\frac{dx}{y-x} = \frac{dz}{x-y}$$

$$\int dx = -\int dz \Rightarrow x = -z + C_1 \Rightarrow \boxed{C_1 = x+z}$$

$$\frac{dy}{C_1+y} = \frac{dx}{y-x}$$

$$y\,dy - x\,dy = C_1 dx + y\,dx$$

$$d(xy) = d\left(C_1 x + \frac{1}{2}y^2\right)$$

$$xy = -C_1 x + \frac{y^2}{2} + C_2$$

$$C_2 = xy - \frac{y^2}{2} + x^2 + xz$$

№1151

$$\frac{dx}{y-u} = \frac{dy}{z-x} = \frac{dz}{u-y} = \frac{du}{x-z}$$

$$dx = -dz \Rightarrow x = -z + C_1, \Rightarrow \boxed{C_1 = x+z}$$

$$dy = -du \Rightarrow \boxed{C_2 = y+u}$$

$$\frac{d(x-z)}{2(y-u)} = \frac{d(y-u)}{2(z-x)}$$

$$-\int (x-z)\,d(x-z) = \int (y-u)\,d(y-u)$$

$$-\frac{(x-z)^2}{2} = \frac{(y-u)^2}{2} + C_3 \Rightarrow \boxed{C_3 = \frac{(y-u)^2 + (x-z)^2}{2}}$$

Ответ: $\begin{cases} C_1 = x+z \\ C_2 = y+u \\ C_3 = \frac{(y-u)^2+(x-z)^2}{2} \end{cases}$

№1156

$$\frac{dx}{x+y^2+z^2} = \frac{2y\,dy}{y} = \frac{2z\,dz}{z}$$

$$\int \frac{dy}{y} = \int \frac{dz}{z} \Rightarrow y = C_1 z \Rightarrow \boxed{C_1 = \frac{y}{z}}$$

$$\frac{d(x-y^2-z^2)}{x-y^2-z^2} = \frac{dy}{y} \Rightarrow \boxed{\frac{x-y^2-z^2}{y} = C_2}$$

№1157

$$\frac{dx}{xy+xz} = \frac{dy}{z^2-zy} = \frac{dz}{y^2-yz}$$

$$\frac{dy}{z} = -\frac{dz}{y} \Leftrightarrow y^2 = -z^2 + C_1 \Rightarrow \boxed{C_1 = y^2+z^2}$$

$\Phi[y] = \int F(x,y,y')dx, \quad y(x_0) = y_0, y(x_1) = y_1$

$$\frac{dy-dz}{(y+z)(y+z)} = \frac{dx}{x(y+z)}$$

$$-\int\frac{d(y-z)}{y-z} = \int\frac{dx}{x} \Rightarrow \boxed{x(y-z)=C_2}$$

Д/з. 1142, 1144, 1146, 1150–1159, 1162, 1164

№1142

$$y' = \frac{y^2}{z-z}$$

$$z' = y+1$$

№1144

$$\begin{cases} y' = y^2 z \\ z' = \frac{z}{x} - yz^2 \end{cases}$$

$$z = \frac{y'}{y^2} \Rightarrow \frac{y''y^2 - 2y(y')^2}{y^4} = \frac{y'}{y^2 x} - \frac{y(y')^2}{y^4} \Leftrightarrow$$

$$\Leftrightarrow \frac{y''y - 2(y')^2}{y^3} = \frac{yy'}{y^3 x} - \frac{(y')^2}{y^3}$$

$$y''y - (y')^2 = \frac{yy'}{x}$$

$$y' = y\,p(x), \quad y'' = y'p + yp' = yp^2 + yp'$$

$$xy\,y'' - (y')^2 x = yy'$$

$$x\cancel{y^2}(p^2+p') - x\cancel{y^2}p^2 = \cancel{y^2}p \qquad \begin{cases} y=0 \text{ — решение} \\ z = Cx \end{cases}$$

$$xp' = p \Rightarrow \frac{dp}{dx} = \frac{p}{x} \qquad \int\frac{dp}{p} = \int\frac{dx}{x} \Rightarrow p = Cx$$

$$\frac{y'}{y} = Cx \qquad \int\frac{dy}{y} = \int Cx\,dx \Rightarrow y = C_1 e^{Cx^2}$$

$$z = \frac{y'}{y^2} = \frac{2Cx}{C_1 e^{Cx^2}}$$

Ответ: $\begin{cases} y = C_1 e^{Cx^2} \\ z = \frac{2Cx}{C_1 e^{Cx^2}} \end{cases}$ $\begin{cases} y = 0 \\ z = Cx \end{cases}$

№1146

$$\frac{dx}{2y-z} = \frac{dy}{y} = \frac{dz}{z}$$

$$\frac{dy}{y} = \frac{dz}{z} \qquad \frac{d(2y-z)}{2y-z} = \frac{dx}{2y-z}$$

$$y = Cz \qquad 2y - z = x + C_1$$

$$C = \frac{y}{z} \qquad C_1 = 2y - z - x$$

Ответ: $C = \frac{y}{z}$ $C_1 = 2y - z - x$

№1150

$$\frac{dx}{z} = \frac{dy}{u} = \frac{dz}{x} = \frac{du}{y}$$

$$\frac{dx}{z} = \frac{dz}{x} \qquad \frac{dy}{u} = \frac{du}{y} \qquad \frac{d(x+z)}{x+z} = \frac{d(u+y)}{u+y}$$

$$x^2 = z^2 + C \qquad u^2 = y^2 + C_1 \qquad x + z = C_2(u+y)$$

Ответ: $C = x^2 - z^2$ $C_1 = u^2 - y^2$ $C_2 = \frac{x+z}{u+y}$

№1152

$$\frac{dx}{z} = \frac{dy}{xz} = \frac{dz}{y}$$

$$\int x\,dx = \int dy$$

$$\frac{x^2}{2} = y + C$$

$$C = \frac{x^2}{2} - y$$

$$\frac{dx}{z} = \frac{dz}{y}$$

$$\frac{dx}{z} = \frac{2dz}{x^2 - 2C}$$

$$\int (x^2 - 2C)\,dx = \int 2z\,dz$$

$$\frac{x^3}{3} - 2Cx = z^2 + C_1$$

Ответ: $C = \frac{x^2}{2} - y$, $C_1 = \frac{x^3}{3} - x^3 + 2xy - z^2$

№1153

$$\frac{dx}{x^2 - y^2} = \frac{dy}{z} = -\frac{dz}{y}$$

$$\frac{dy}{z} = -\frac{dz}{y}$$

$$\int y\,dy = -\int z\,dz$$

$$y^2 = -z^2 + C$$

$$C = y^2 + z^2$$

$$\frac{dx}{z^2 - y^2} = \frac{z\,dy + y\,dz}{z^2 - y^2} \qquad \int dx = \int d(yz) \Rightarrow x - yz = C_1$$

Ответ: $C = y^2 + z^2$ $\quad C_1 = x - yz$

№1154

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{xy + z}$$

$$\frac{dx}{x} = \frac{dy}{y} \qquad \frac{dz}{y} = Cy^2 + z$$

$$x = Cy$$

$$C = x/y$$

$$y\frac{dz}{dy} = Cy^2 + z$$

$$z' = Cy + \frac{z}{y}$$

I. $z' = \frac{z}{y} \Rightarrow z = C_1 y$

II. $C_1 = C_1(y)$ $\quad C_1'y = Cy$

$$C_1' = C \Rightarrow C_1 = Cy + C_2$$

$$z = (Cy + C_2)y = xy + C_2 y$$

Ответ: $C = x/y$ $\quad C_2 = z/y - x$

№1155

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy\sqrt{z^2+1}}$$

$$\int \frac{dx}{x} = \int \frac{dy}{y} \qquad \frac{y\,dx + x\,dy}{2xyz} = \frac{dz}{xy\sqrt{z^2+1}}$$

$$x = Cy \qquad \int d(xy) = \int \frac{dz^2}{\sqrt{z^2+1}}$$

$$xy = 2\sqrt{z^2+1} + C_1$$

Ответ: $C = x/y$ $\quad C_1 = xy - 2\sqrt{z^2+1}$

№1158

$$-\frac{dx}{x^2} = \frac{dy}{xy - 2z^2} = \frac{dz}{xz}$$

$$\Phi[y] = \int_{x_0}^{x_1} F(x, y, y')\,dx, \quad y(x_0) = y_0,\ y(x_1) = y_1$$

$\int \frac{}{x} = \int \frac{}{z}$ $\qquad x^2 = xy - 2z^2$

$z = C/x$ $\qquad -xy\,dx + 2z^2 dx = x^2 dy$

$\cancel{x}\ d(xy) = 2z^2 dx$

$\int d(xy) = \int \frac{2C^2}{x^3} dx$

$xy = 2C^2\left(-\frac{1}{2x^2}\right) + C_1$

$xy = -\cancel{z}\, z^2 + \cancel{z}\, C_1$

Ответ: $C = zx \qquad C_1 = \frac{xy + z^2}{\cancel{2}}$

№1159

$$\frac{dx}{x(z-y)} = \frac{dy}{y(y-x)} = \frac{dz}{y^2 - xz}$$

$$\frac{d(x+z)}{y(y-x)} = \frac{dy}{y(y-x)}$$

$\int d(x+z) = \int dy$

$x + z - y = C$

$$\frac{dx}{x(C-x)} = \frac{dy}{y(y-x)}$$

$$y' + \frac{y}{C-x} = \frac{y^2}{x(C-x)}$$

$$\frac{y'}{y^2} + \frac{1}{y(C-x)} - \frac{1}{x(C-x)} = 0$$

$t = \frac{1}{y}, \quad t' = -\frac{y'}{y^2}$

$$-t' + \frac{t}{C-x} - \frac{1}{x(C-x)} = 0$$

$t'(C-x)x - tx + 1 = 0$

I. $t'(C-x)x = tx$

$$\int \frac{dt}{t} = \int \frac{\cancel{x}\,dx}{C-x}$$

$$t = \frac{C_1}{C-x}$$

II. $C_1 = C_1(x)$

$$\frac{C_1'(C-x) + C_1}{(C-x)^2}(C-x)x - tx + 1 = 0$$

$$C_1'x + \frac{C_1 \cancel{x}}{C-\cancel{x}} - \frac{C_1 x}{C-x} = -1$$

$C_1' = -\frac{1}{x} \Rightarrow C_1 = -\ln|x| + C_2$

$$y = \frac{1}{t} = \frac{C-x}{C_2 - \ln|x|} = \frac{z-y}{C_2 - \ln|x|}$$

Ответ: $C = x + z - y \qquad C_2 = \ln|x| + \frac{z}{y} - 1$

№1162

$\dot{x} = xy, \quad \dot{y} = x^2 + y^2, \quad \varphi_1 = x\ln y - x^2 y; \quad \varphi_2 = \frac{y^2}{x}z - 2\ln x$

$$\dot{\varphi}_1 = \dot{x}\ln y + x\frac{1}{y}\dot{y} - 2xy\dot{x} - x^2\dot{y} = xy\ln y + \frac{x}{y(x^2+y^2)} -$$

$$-2x^2y^2 - x^2(x^2+y^2) \neq 0$$

$$\dot{\varphi}_2 = \frac{2y\dot{y}}{x^2} - 2\frac{y^2 \dot{x}}{x^3} - \frac{2\dot{x}}{x} = \frac{2y(x^2+y^2)}{x^2} - \frac{2y^3 x}{x^3} - 2y = 0$$

Ответ: 1) нет 2) да

$\Phi[y] = \int^{x_1} F(x, y, y')dx, \quad y(x_0) = y_0, \ y(x_1) = y_1$

$$\frac{x+y}{z+x}=C_1, \quad \frac{z-y}{x+y}=C_2, \quad \frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}$$

$$\begin{vmatrix} \frac{\partial C_1}{\partial x} & \frac{\partial C_2}{\partial x} \\ \frac{\partial C_1}{\partial y} & \frac{\partial C_2}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{z+x-x-y}{(z+x)^2} & -\frac{z-y}{(x+y)^2} \\ \frac{1}{z+x} & \frac{-x-y-z+y}{(x+y)^2} \end{vmatrix} =$$

$$= -\frac{(z-y)(z+x)}{(z+x)^2(x+y)^2} + \frac{z-y}{(z+x)(x+y)^2} = 0$$

$$\begin{vmatrix} \frac{\partial C_1}{\partial y} & \frac{\partial C_2}{\partial y} \\ \frac{\partial C_1}{\partial z} & \frac{\partial C_2}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{1}{z+x} & \frac{-x-y-z+y}{(x+y)^2} \\ -\frac{x+y}{(z+x)^2} & \frac{1}{x+y} \end{vmatrix} =$$

$$= \frac{1}{(z+x)(x+y)} - \frac{(x+y)(x+z)}{(x+y)^2(x+z)^2} = 0$$

$$\begin{vmatrix} \frac{\partial C_1}{\partial x} & \frac{\partial C_2}{\partial x} \\ \frac{\partial C_1}{\partial z} & \frac{\partial C_2}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{z-y}{(x+z)^2} & -\frac{z-y}{(x+y)^2} \\ -\frac{(x+y)}{(x+z)^2} & \frac{1}{x+y} \end{vmatrix} =$$

$$= \frac{z-y}{(x+z)^2(x+y)} - \frac{(z-y)(x+y)}{(x+z)^2(x+y)^2} = 0 \Rightarrow$$ перв. инт-зав.

$z(x,y)$

$$z_x = \frac{\partial z}{\partial x}, \quad z_y = \frac{\partial z}{\partial y} \Rightarrow a_1\frac{\partial z}{\partial x} + a_2\frac{\partial z}{\partial y} = b$$

$$a_i = a_i(x,y,z)$$

$$b = b(x,y,z)$$

$$\frac{dx}{a_1} = \frac{dy}{a_2} = \frac{dz}{b} \Rightarrow C_1 = \varphi_1(x,y,z), \quad C_2 = \varphi_2(x,y,z)$$

Ответ: $F(C_1, C_2) = 0 \Rightarrow F(\varphi_1(x,y,z), \varphi_2(x,y,z)) = 0$

II сл. Если иск. функ. z входит только в один перв. инт-л.

$C_1 = \varphi_1(x,y,z)$, $C_2 = \varphi_2(x,y) \Rightarrow C_1 = f(C_2)$, $\varphi_1(x,y,z) = f(\varphi_2(x,y))$

№ 1167

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$$

$$\frac{dx}{y} = -\frac{dy}{x} = \frac{dz}{0}$$

1) $dz = 0 \Rightarrow z = C_1$

2) $\frac{dx}{y} = -\frac{dy}{x} \Rightarrow \int x\,dx = \int -y\,dy \to$

$$x^2 + y^2 = C_2$$

Ответ: 1) $F(z, x^2+y^2) = 0$

2) $C_1 = f(C_2) \to z = f(x^2+y^2)$

$$(x-z)\frac{\partial u}{\partial x}+(y-z)\frac{\partial u}{\partial y}+2z\frac{\partial u}{\partial z}=0$$

$$\frac{dx}{x-z}=\frac{dy}{y-z}=\frac{dz}{2z}=\frac{du}{0}$$

1) $du=0 \Rightarrow u=C_1$

2) $\int\frac{d(x-y)}{x-y}=\int\frac{dz}{2z}$

$x-y=C_2\sqrt{z} \Rightarrow C_2=\frac{(x-y)^2}{z}$

3) $\cancel{\frac{dx}{x-z}=\frac{dz}{2z}}$ $\quad \frac{d(x+z)}{x+z}=\frac{dz}{2z}$

$$C_3=\frac{(x+z)^2}{z}$$

Ответ: $C_1=f(C_2,C_3) \Rightarrow u=f\left(\frac{(x-y)^2}{z},\frac{(x+z)^2}{z}\right)$

№ 1174

$$xy\frac{\partial z}{\partial x}-x^2\frac{\partial z}{\partial y}=yz$$

$$\frac{dx}{xy}=\frac{dy}{-x^2}=\frac{dz}{yz}$$

1) $\frac{dx}{xy}=\frac{dz}{yz}$

$C_1x=z \Rightarrow C_1=\frac{z}{x}$

2) $\frac{dx}{xy}=-\frac{dy}{x^2}$ $\quad -\int x\,dx=\int y\,dy \Rightarrow x^2+y^2=C_2$

Ответ: $C_1=f(C_2) \Rightarrow \frac{z}{x}=f(x^2+y^2) \Rightarrow z=x\,f(x^2+y^2)$

№ 1189

$$x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=0 \qquad z=2x \text{ при } y=1$$

$$\frac{dx}{x}=-\frac{dy}{y}=\frac{dz}{0}$$

1) $dz=0 \Rightarrow z=C_1$

2) $\int\frac{dx}{x}=-\int\frac{dy}{y} \Rightarrow yx=C_2$

$z=f(xy) \qquad y=1 \Rightarrow z=2x$

$2x=f(x)$

Ответ: $z=2xy$

№ 1195

$$x\frac{\partial z}{\partial x}-2y\frac{\partial z}{\partial y}=x^2+y^2; \quad y=1,\ z=x^2$$

$$\frac{dx}{x}=-\frac{dy}{2y}=\frac{dz}{x^2+y^2}$$

1) $\int\frac{dx}{x}=-\frac{1}{2}\int\frac{dy}{y} \Rightarrow x^2=\frac{C_1}{y} \Rightarrow C_1=x^2y$

2) $\frac{d(x^2-\frac{y^2}{2})}{2x^2+2y^2}=\frac{dz}{x^2+y^2}$

$\int\frac{d(x^2-y^2/2)}{2}=\int dz \Rightarrow z=\frac{1}{2}(x^2-\frac{y^2}{2})+\frac{C_2}{4} \Rightarrow C_2=4z-2x^2+y^2$

$\Phi[y]=\int^{x_1}F(x,y,y')dx, \quad y(x_0)=y_0,\ y(x_1)=y_1$

$C_2 = f(C_1)$

$4z - 2x^2 + y^2 = f(x^2 y)$

$y = 1,\ z = x^2$

$4x^2 - 2x^2 + 1 = f(x^2)$

$f(t) = 2t + 1$

Ответ: $4z - 2x^2 + y^2 = 2x^2 y + 1 \Rightarrow 4z = 2x^2 - y^2 + 2x^2 y + 1$

II сп.

$$\begin{cases} C_1 = x^2 y \\ C_2 = 4z - 2x^2 + y^2 \\ y = 1 \\ z = x^2 \end{cases} \Rightarrow \begin{cases} C_1 = x^2 \\ C_2 = 2x^2 + 1 \\ \cancel{y = 1} \\ \cancel{z = x^2} \end{cases} \Rightarrow C_2 = 2C_1 + 1$$

№ 1198

$$x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} = z^2 (x - 3y) \quad ; \quad x = 1,\ yz + 1 = 0$$

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{z^2(x-3y)}$$

1) $\int \frac{dx}{x} = -\int \frac{dy}{y} \Rightarrow xy = C_1$

2) $\frac{d(x+3y)}{\cancel{x-3y}} = \frac{dz}{z^2\cancel{(x-3y)}}$

$x + 3y = -\frac{1}{z} + C_2$

$$\begin{cases} \ldots \\ C_2 = x + 3y + \frac{1}{z} \\ x = 1 \\ yz = -1 \end{cases} \Leftrightarrow \begin{cases} \ldots \\ C_2 = 1 + \frac{3}{z} + \frac{1}{z} \end{cases} \Leftrightarrow \begin{cases} \ldots \\ C_2 = 1 + 2y \end{cases}$$

$$\boxed{C_2 = 1 + 2C_1}$$

Ответ: $x + 3y + \frac{1}{z} = 1 + 2xy$

№ 1203

$$(y - z) \frac{\partial z}{\partial x} + (z - x) \frac{\partial z}{\partial y} = x - y \ ; \ z = y = -x$$

$$\frac{dx}{y - z} = \frac{dy}{z - x} = \frac{dz}{x - y}$$

1) $\frac{d(x+y)}{\cancel{y-x}} = \frac{dz}{\cancel{x-y}}$

$\int d(x+y) = -\int dz \Rightarrow x + y + z = C_1$

2) $\frac{x\,dx + y\,dy}{\cancel{xy} - xz + yz - \cancel{xy}} = \frac{dz}{x - y}$

$\int d\left(\frac{x^2}{2} + \frac{y^2}{2}\right) = -\int z\,dz$

$x^2 + y^2 + z^2 = C_2$

$$\begin{cases} C_1 = x + y + z \\ C_2 = x^2 + y^2 + z^2 \\ z = y \\ x = -y \end{cases} \Leftrightarrow \begin{cases} C_1 = y \\ C_2 = 3y^2 \end{cases} \Rightarrow C_2 = 3C_1^2$$

Ответ: $x^2 + y^2 + z^2 = 3(x + y + z)^2$

$\Phi[y] = \int_{x_0}^{x_1} F(x, y, y')\,dx, \quad y(x_0) = y_0,\ y(x_1) = y_1$

№1210

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2xy \qquad y=x,\ z=x^2$$

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{2xy}$$

1) $\int\frac{dx}{x}=\int\frac{dy}{y}$

$$C_1=\frac{x}{y}$$

2) $\frac{dxy}{2xy}=\frac{dz}{2xy}$

$$-xy+z=C_2$$

$$\begin{cases}C_1=x/y\\ C_2=xy-z\\ y=x\\ z=x^2\end{cases} \Rightarrow \begin{cases}C_1=1\\ C_2=0\end{cases}$$

$$C_2=f(C_1)$$

$$-xy+z=f(x/y)$$

$$z=xy+f(x/y)$$

$$x^2=x^2+f(1)$$

$$f(1)=0$$

Ответ: $z=xy+f(x/y)$, где $f(t)$ - ∀ дифф. - функц., $f(1)=0$

$$(y+z)\frac{\partial u}{\partial x}+(z+x)\frac{\partial u}{\partial y}+(x+y)\frac{\partial u}{\partial z}=u$$

$$\frac{dx}{y+z}=\frac{dy}{z+x}=\frac{dz}{x+y}=\frac{du}{u}$$

1) $\frac{d(x+y+z)}{2(x+y+z)}=\frac{du}{u}$

$$\frac{u^2}{(x+y+z)}=C_1$$

2) $\frac{d(x-y)}{y-x}=\frac{du}{u}$

$$C_2=u(x-y)$$

3) $\int\frac{d(x-z)}{z-x}=\int\frac{du}{u}$

$$C_3=u(x-z)$$

Ответ: $F\left(\frac{u^2}{x+y+z},\ u(x-y),\ u(x-z)\right)=0$

~~№1188~~

$\cancel{(u-x)\frac{\partial u}{\partial x}+(u-y)}$

№1194

$$y^2\frac{\partial z}{\partial x}+xy\frac{\partial z}{\partial y}=x \qquad x=0,\ z=y^2$$

$$\frac{dx}{y^2}=\frac{dy}{xy}=\frac{dz}{x}$$

1) $\int x\,dx=\int y\,dy \qquad x^2-y^2=C_1$

2) $\int\frac{dy}{y}=\int dz$

$$y=C_2e^z \Rightarrow C_2=\frac{y}{e^z}$$

$\Phi[y]=\int_{x_0}^{x_1}F(x,y,y')dx,\quad y(x_0)=y_0,\ y(x_1)=y_1$

$$\begin{cases} c_2 = y e^{-z} \\ x = 0 \\ z = y^2 \end{cases} \Rightarrow c_2 = y e^{-y^2}$$

$$c_2 = \pm\sqrt{-c_1}\, e^{c_1}$$

$$y e^{-z} = \pm\sqrt{y^2 - x^2}\, e^{x^2 - y^2}$$

Ответ: $y = \pm\sqrt{y^2 - x^2}\, e^{x^2 - y^2 + z}$

Д/з. | 1168, 1169, 1178, 1182, 1184, 1191, 1193, 1200, 1202, 1204, 1207, 1208

№ 1168

$$(x+2y)\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 0$$

$$\frac{dx}{x+2y} = \frac{dy}{-y} = \frac{dz}{0}$$

1) $z = c_1$

2) $\frac{dx}{x+2y} = -\frac{dy}{y}$

$y\,dx + x\,dy + dy^2 = 0 \Rightarrow d(xy + y^2) = 0$

$xy + y^2 = c_2$

$c_1 = f(c_2)$

Ответ: $z = f(xy + y^2)$

№ 1169

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} = \frac{du}{0}$$

1) $u = c_1$

2) $\int\frac{dx}{x} = \int\frac{dy}{y} \Rightarrow \frac{x}{y} = c_2$

3) $\int\frac{dx}{x} = \int\frac{dz}{z} \Rightarrow \frac{x}{z} = c_3$

Ответ: $u = f\left(\frac{x}{y}, \frac{x}{z}\right)$

№ 1178

$$x^2 z\frac{\partial z}{\partial x} + y^2 z\frac{\partial z}{\partial y} = x + y$$

$$\frac{dx}{x^2 z} = \frac{dy}{y^2 z} = \frac{dz}{x+y}$$

1) $\int\frac{dx}{x^2} = \int\frac{dy}{y^2} \Rightarrow -\frac{1}{x} = -\frac{1}{y} + c_1 \Rightarrow c_1 = \frac{1}{x} - \frac{1}{y}$

2) $\frac{y\,dx + x\,dy}{xyz(\cancel{x+y})} = \frac{dz}{\cancel{x+y}} \Rightarrow \int\frac{d(xy)}{xy} = \int z\,dz$

$$\ln|xy| = \frac{z^2}{2} + c_2$$

Ответ: $F\left(\frac{1}{x} - \frac{1}{y}, \ln|xy| - \frac{z^2}{2}\right) = 0$

$\Phi[y] = \int^{x_1} F(x, y, y')\,dx, \quad y(x_0) = y_0,\ y(x_1) = y_1$

№1182

$$y\frac{\partial z}{\partial x} + z\frac{\partial z}{\partial y} = \frac{y}{x}$$

$$\frac{dx}{y} = \frac{dy}{z} = \frac{dz}{y/x}$$

1) $\frac{dx}{y} = \frac{x\,dz}{y} \Rightarrow \int\frac{dx}{x} = \int dz$

$$\ln|x| + C_1 = z \Rightarrow C_1 = z - \ln|x|$$

2) $\frac{z\,dx + x\,dz}{zy + y} = \frac{dy}{z}$

$$\frac{d(xz)}{y(z+1)} = \frac{dy}{z}$$

$$\frac{d(xz - x)}{yz} = \frac{y\,dy}{yz}$$

$$xz - x = \frac{y^2}{2} + \frac{C_2}{2}$$

$$2x(z-1) - y^2 = C_2$$

Ответ: $F(z - \ln|x|,\ 2x(z-1) - y^2) = 0$

№1184

$$(x+z)\frac{\partial z}{\partial x} + (y+z)\frac{\partial z}{\partial y} = x + y$$

$$\frac{dx}{x+z} = \frac{dy}{y+z} = \frac{dz}{x+y}$$

1) $\int\frac{d(x+y+z)}{2(x+y+z)} = \int\frac{\dots}{x-y}$

$$\frac{x+y+z}{(x-y)^2} = C_1$$

2) $\frac{d(x-z)}{z-y} = \frac{d(y-z)}{z-x}$

$$\int (x-z)\,d(x-z) = \int (y-z)\,d(y-z)$$

$$(x-z)^2 = (y-z)^2 + C_2 \Rightarrow C_2 = (x-y)(x+y-2z)$$

Ответ: $F\left(\frac{x+y+z}{(x-y)^2},\ (x-y)(x+y-2z)\right) = 0$

№1191

$2\sqrt{x}\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 0$ $\quad z = y^2$ при $x = 1$ $\quad \frac{dx}{2\sqrt{x}} = -\frac{dy}{y} = \frac{dz}{0}$

1) $\frac{dx}{2\sqrt{x}} = -\frac{dy}{y} \Rightarrow d\sqrt{x} = -d\ln|y|$

$$\ln|y| + \sqrt{x} = C_1$$

2) $z = C_2$

$$z = f(\ln|y| + \sqrt{x}) \Rightarrow y^2 = f(\ln|y| + 1) \Rightarrow$$

$$f(a) = e^{2a-2} \Rightarrow z = e^{\ln y^2 + 2\sqrt{x} - 2} = y^2 e^{2\sqrt{x} - 2}$$

Ответ: $z = y^2 e^{2\sqrt{x} - 2}$

$\Phi[y] = \int^{x_1} F(x, y, y')\,dx, \quad y(x_0) = y_0, y(x_1) = y_1$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + xy\frac{\partial u}{\partial z} = 0 \qquad u = x^2+y^2 \text{ при } z=0$$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{xy} = \frac{du}{0}$$

1) $u = C_1$

2) $\int\frac{dx}{x} = \int\frac{dy}{y} \Rightarrow \frac{x}{y} = C_2$

3) $\frac{y\,dx + x\,dy}{2xy} = \frac{dz}{xy}$

$\int d(xy) = \int 2dz \Rightarrow xy = 2z + C_3 \Rightarrow C_3 = xy - 2z$

$$\begin{cases} C_1 = u \\ C_2 = x/y \\ C_3 = xy - 2z \end{cases} \Rightarrow \begin{cases} C_1 = x^2+y^2 \\ C_2 = x/y \\ C_3 = xy \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} yC_2 = x \\ C_1 = y^2(C_2^2+1) \\ C_3 = y^2 C_2 \end{cases} \Rightarrow C_1 = \frac{C_3}{C_2}(C_2^2+1)$$

$$u = \frac{y(xy-2z)}{x}\left(\frac{x^2}{y^2}+1\right) = (xy-2z)\left(\frac{x}{y}+\frac{y}{x}\right)$$

Ответ: $u = (xy-2z)\left(\frac{x}{y}+\frac{y}{x}\right)$

№1200

$$yz\frac{\partial z}{\partial x} + xz\frac{\partial z}{\partial y} = xy \qquad x = a,\ y^2+z^2 = a^2$$

$$\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy}$$

1) $\int x\,dx = \int y\,dy \Rightarrow x^2 - y^2 = C_1$

2) $\cancel{\frac{x\,dx + y\,dy}{2xyz} = \frac{dz}{xy}}$

$\cancel{\int d(x^2+y^2) = \int 4\,dz \Rightarrow x^2+y^2-4z = C_2}$

$\cancel{\begin{cases} C_1 = x^2-y^2 \\ C_2 = x^2+y^2-4z \end{cases} \Rightarrow \begin{cases} C_1 = a^2-y^2 \\ C_2 = a^2+y^2-4z \\ a^2 = y^2+z^2 \end{cases} \Rightarrow \begin{cases} C_1 = z^2 \\ C_2 = 2y^2+z^2-4z \end{cases}}$

2) $\int y\,dy = \int z\,dz \Rightarrow y^2 - z^2 = C_2$

$$\begin{cases} C_1 = x^2-y^2 \\ C_2 = y^2-z^2 \end{cases} \Rightarrow \begin{cases} C_1 = a^2-y^2 \\ C_2 = 2y^2-a^2 \end{cases} \Rightarrow C_2 = -2C_1 + a^2$$

$$y^2 - z^2 = -2x^2 + 2y^2 + a^2$$

Ответ: $a^2 = 2x^2 - y^2 - z^2$

№1202

$$z\frac{\partial z}{\partial x} + (z^2-x^2)\frac{\partial z}{\partial y} + x = 0 \qquad y = x^2,\ z = 2x$$

$$\frac{dx}{z} = \frac{dy}{z^2-x^2} = \frac{dz}{-x}$$

1) $\int -x\,dx = \int z\,dz \Rightarrow z^2 + x^2 = C_1$

2) $\frac{z\,dx + x\,dz}{z^2-x^2} = \frac{dy}{z^2-x^2} \Rightarrow \int d(xz) = \int dy \Rightarrow xz - y = C_2$

$\Phi[y] = \int_{x_0}^{x_1} F(x,y,y')\,dx, \quad y(x_0) = y_0,\ y(x_1) = y_1$

$$\begin{cases} C_1 = z^2+x^2 \\ C_2 = xz - y \end{cases} \Rightarrow \begin{cases} C_1 = x^2 + 4x^2 \\ C_2 = 2x^2 - x^2 \end{cases} \Rightarrow \begin{cases} C_1 = 5x^2 \\ C_2 = x^2 \end{cases}$$

$C_1 = 5C_2$

Ответ: $x^2+z^2 = 5(xz-y)$

№1204

$$x\frac{\partial z}{\partial x} + (xz+y)\frac{\partial z}{\partial y} = z \qquad x+y=2z,\ xz=1$$

$$\frac{dx}{x} = \frac{dy}{xz+y} = \frac{dz}{z}$$

1) $\int\frac{dx}{x} = \int\frac{dz}{z} \Rightarrow \frac{z}{x} = C_1$

2) $\frac{\cancel{x}dy - z\,dx - x\,dz}{\cancel{x}y - xz} = \frac{dx}{x}$

$$\int\frac{d(y-xz)}{y-xz} = \int\frac{dx}{x}$$

$$C_2 = \frac{y-xz}{x}$$

$$\begin{cases} C_1 = \frac{z}{x} \\ C_2 = \frac{y}{x} - z \end{cases} \Leftrightarrow \begin{cases} xC_1 = z \\ xC_2 = y - xz \end{cases} \Rightarrow \begin{cases} xC_1 = x+y \\ xC_2 = y-1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \cancel{C_1 = \frac{1}{2} + \frac{y}{2x}} \\ \cancel{C_2 = \frac{y}{x} - \frac{1}{x}} \end{cases} \Leftrightarrow \begin{cases} \cancel{C_1 = 1 + \frac{y}{x}} \\ \cancel{C_2 = \frac{y}{x} - \frac{1}{2}(x+y)} \end{cases} \Rightarrow \cancel{C_2 = 2C_1 - 1 - xC_1}$$

$$\boxed{x+y = 2z \Leftrightarrow \frac{1}{z} + y = 2z}$$

$$\Rightarrow \begin{cases} C_1 = z^2 \\ C_2 = yz - z \end{cases} \Rightarrow \begin{cases} C_1 = z^2 \\ C_2 = -1 + 2z^2 - z \end{cases}$$

$C_2 = -1 + 2C_1 - \sqrt{C_1}$

$$\frac{y}{x} - z = -1 + 2\frac{z}{x} - \sqrt{\frac{z}{x}} \Leftrightarrow y - zx = -x + 2z - \sqrt{xz} \Leftrightarrow$$

$$\Leftrightarrow -2z + x + y = zx - \sqrt{xz} \Rightarrow xz = (zx + 2z - y - x)^2$$

Ответ: $xz = (xz + 2z - y - x)^2$

№1207

$$(y+2z^2)\frac{\partial z}{\partial x} - 2x^2z\frac{\partial z}{\partial y} = x^2 \qquad x = z,\ y = x^2$$

$$\frac{dx}{y+2z^2} = \frac{dy}{-2x^2z} = \frac{dz}{x^2}$$

1) $\int dy = \int -2z\,dz \Rightarrow y + z^2 = C_1$

2) $\frac{dx}{y+2z^2} = \frac{dz}{x^2} \Leftrightarrow \frac{dx}{C_1+z^2} = \frac{dz}{x^2}$

$$\int x^2dx = \int (C_1+z^2)dz \Rightarrow \frac{x^3}{3} = C_1z + \frac{z^3}{3} + \frac{C_2}{3}$$

$C_2 = x^3 - z^3 - 3(z^2+y)z$

$$\begin{cases} C_1 = y+z^2 \\ C_2 = x^3 - z^3 - 3z(z^2+y) \end{cases} \Rightarrow \begin{cases} C_1 = 2x^2 \\ C_2 = \cancel{x^3} - \cancel{x^3} - 3x\cdot 2x^2 \end{cases} \Leftrightarrow \begin{cases} C_1 = 2x^2 \\ C_2 = -6x^3 \end{cases}$$

$$\Rightarrow C_2 = -6\left(\frac{C_1}{2}\right)^{3/2} \Leftrightarrow C_2^2 = 36\cdot\frac{C_1^3}{8} \Leftrightarrow 2C_2^2 = 9C_1^3$$

$$\Phi[y] = \int_{x_0}^{x_1} F(x,y,y')dx, \quad y(x_0) = y_0,\ y(x_1) = y_1$$

Ответ: $2(x^3 - z^2 - 3z(z^2+y))^2 = -9(z^2+y)^3$

№1208

$$(x-z)\frac{\partial z}{\partial x} + (y-z)\frac{\partial z}{\partial y} = 2z \qquad x-y=2,\ z+2x=1$$

$$(y=x-2,\ z=1-2x)$$

$$\frac{dx}{x-z} = \frac{dy}{y-z} = \frac{dz}{2z}$$

1) $\int \frac{d(x-y)}{x-y} = \int \frac{dz}{2z}$

$$C_1 = \frac{z}{(x-y)^2}$$

2) $\int \frac{d(y+z)}{y+z} = \int \frac{d(x+z)}{x+z}$

$$\frac{y+z}{x+z} = C_2$$

$$\begin{cases} C_1 = \frac{z}{(x-y)^2} \\ C_2 = \frac{y+z}{x+z} \end{cases} \Rightarrow \begin{cases} C_1 = \frac{1-2x}{4} \\ C_2 = \frac{x-2+1-2x}{x+1-2x} \end{cases} \Leftrightarrow \begin{cases} 4C_1 = 1-2x \\ C_2 = \frac{-1-x}{1-x} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = \frac{1}{2} - 2C_1 \\ C_2 = \frac{-1-\frac{1}{2}+2C_1}{1-\frac{1}{2}+2C_1} \end{cases} \Rightarrow C_2 = \frac{2C_1 - \frac{3}{2}}{2C_1 + \frac{1}{2}} = \frac{4C_1 - 3}{4C_1 + 1}$$

$$\frac{y+z}{x+z} = \frac{\frac{4z}{(x-y)^2} - 3}{\frac{4z}{(x-y)^2} + 1} = \frac{4z - 3(x-y)^2}{4z + (x-y)^2} \Rightarrow$$

$$\Rightarrow 4z(x+z) - 3(x-y)^2(x+z) = 4z(y+z) + (x-y)^2(y+z) \Leftrightarrow$$

$$\Leftrightarrow 4z(x-y) - (x-y)^2(3x+y+4z) = 0 \Leftrightarrow$$

$$\Leftrightarrow (x-y)(3x+y+4z) = 4z$$

Ответ: $4z = (x-y)(3x+y+4z)$

Краснов „Вар. исч."

$$J[y(x)] = \int_{x_0}^{x_1} F(x, y, y')\,dx \qquad y(x_0) = y_0 \quad y(x_1) = y_1$$

$$\Rightarrow \begin{cases} F'_y - \frac{d}{dx} F'_{y'} = 0 \\ y(x_0) = y_0 \\ y(x_1) = y_1 \end{cases}$$ — ур-ние Эйлера

$$J[y(x)] = \int_{x_0}^{x_1} F(x, y, y', \dots, y^{(n)})\,dx \Rightarrow$$

$$\Rightarrow F'_y - \frac{d}{dx}F'_{y'} + \frac{d^2}{dx^2}F'_{y''} - \frac{d^3}{dx^3}F'_{y'''} + \dots + \frac{(-1)^n d^n}{dx^n} F'_{y^{(n)}} = 0$$

+ граничные условия

$$J[y(x), z(x)] = \int_{x_0}^{x_1} F(x, y, y', z, z')\,dx \Rightarrow$$

$$\Rightarrow \begin{cases} F'_y - \frac{d}{dx} F'_{y'} = 0 \\ F'_z - \frac{d}{dx} F'_{z'} = 0 \end{cases}$$
+ гр. усл.

$\Phi[y] = \int_{x_0}^{x_1} F(x, y, y')\,dx, \quad y(x_0) = y_0,\ y(x_1) = y_1$

№ 1

$$J[y] = \int_{-1}^{0} (12xy - y'^2)\,dx \qquad y(-1) = 1,\ y(0) = 0$$

$$F_y = 12x \qquad F_{y'} = -2y'$$

$$\begin{cases} 12x + 2y'' = 0 \\ y(-1) = 1 \\ y(0) = 0 \end{cases} \Rightarrow \quad 6x^2 + 2y' = 2C_1 \Rightarrow y' = C_1 - 3x^2$$

$$y = C_1 x - x^3 + C_2$$

$$\begin{cases} C_1 + 1 + C_2 = 1 \\ C_2 = 0 \end{cases} \Rightarrow \begin{cases} C_1 = 0 \\ C_2 = 0 \end{cases} \Rightarrow y = -x^3$$

Ответ: $y = -x^3$

№ 2

$$J[y] = \int_{1}^{3} (3x - y)y\,dx \qquad y(1) = 1 \qquad y(3) = 4{,}5$$

$$F_y = 3x - 2y \qquad F_{y'} = 0$$

$$\begin{cases} 3x - 2y = 0 \\ y(1) = 1 \\ y(3) = 4{,}5 \end{cases} \Rightarrow \quad y = \frac{3}{2}x$$

$$\frac{3}{2}\cdot 1 \neq 1 \qquad \frac{3}{2}\cdot 3 = 4{,}5$$

Ответ: $\varnothing$

$$J[y] = \int_{0}^{2\pi} (y'^2 - y^2)\,dx \qquad y(0) = 1 \qquad y(2\pi) = 1$$

$$F_y = -2y \qquad F_{y'} = 2y'$$

$$\begin{cases} -2y - 2y'' = 0 \\ y(0) = 1 \\ y(2\pi) = 1 \end{cases} \Rightarrow \quad y'' + y = 0 \qquad \lambda^2 + 1 = 0 \qquad \lambda = \pm i$$

$$y = C_1 \cos x + C_2 \sin x$$

$$\begin{cases} C_1 = 1 \\ C_1 = 1 \end{cases}$$

Ответ: $y = \cos x + C \sin x$

№ 4

$$J[y] = \int_{-1}^{1} (y'^2 - 2xy)\,dx \qquad y(-1) = -1 \qquad y(1) = 1$$

$$F_y = -2x \qquad F_{y'} = 2y'$$

$$\begin{cases} -2x - 2y'' = 0 \\ y(-1) = -1 \\ y(1) = 1 \end{cases} \Rightarrow \quad y'' = -x \Rightarrow y' = -\frac{x^2}{2} + C_1 \Rightarrow$$

$$y = -\frac{x^3}{6} + C_1 x + C_2$$

$$\begin{cases} \frac{1}{6} - C_1 + C_2 = -1 \\ -\frac{1}{6} + C_1 + C_2 = 1 \end{cases} \Rightarrow \begin{cases} C_2 = 0 \\ C_1 = \frac{7}{6} \end{cases} \Rightarrow y = -\frac{x^3}{6} + \frac{7}{6}x$$

Ответ: $y = -\frac{x^3}{6} + \frac{7}{6}x$

$$\Phi[y] = \int_{x_0}^{x_1} F(x, y, y')\,dx, \quad y(x_0) = y_0,\ y(x_1) = y_1$$

№ 5

$$J[y] = \int_0^1 (360x^2 y - y''^2)\,dx \qquad y(0)=0 \quad y'(0)=1$$
$$y(1)=0 \quad y'(1)=2{,}5$$

$F_y = 360x^2 \quad F_{y'} = 0 \quad F_{y''} = -2y''$

$$\begin{cases} 360x^2 - 2y'''' = 0 \\ y(0)=0 \quad y'(0)=1 \\ y(1)=0 \quad y'(1)=2{,}5 \end{cases} \qquad y^{IV} = 180x^2 \Rightarrow y''' = 60x^3 + C_1 \Rightarrow$$

$\Rightarrow y'' = 15x^4 + C_1x + C_2 \Rightarrow$

$\Rightarrow y' = 3x^5 + \frac{C_1}{2}x^2 + C_2x + C_3 \Rightarrow$

$\Rightarrow y = \frac{1}{2}x^6 + \frac{C_1}{6}x^3 + \frac{C_2}{2}x^2 + C_3x + C_4$

$$\begin{cases} C_4 = 0 \\ \frac{1}{2} + \frac{C_1}{6} + \frac{C_2}{2} + C_3 = 0 \\ C_3 = 1 \\ 3 + \frac{C_1}{2} + C_2 + 1 = 2{,}5 \end{cases} \Leftrightarrow \begin{cases} C_4 = 0 \\ C_3 = 1 \\ C_1 + 3C_2 + 6 + 3 = 0 \\ C_1 + 2C_2 = -3 \end{cases} \Leftrightarrow \begin{cases} C_4 = 0 \\ C_3 = 1 \\ C_2 = -6 \\ C_1 = 9 \end{cases}$$

Ответ: $y = \frac{x^6}{2} + \frac{3}{2}x^3 - 3x^2 + x$

№ 7

$$\cancel{J[y] = \frac{1}{2}\int_0^1 (y'')^2\,dx} \qquad J[y] = \int_1^2 (y'^2 + z^2 + z'^2)\,dx$$
$$y(1)=1 \quad y(2)=2$$
$$z(1)=0 \quad z(2)=1$$

$F_y = 0 \quad F_{y'} = 2y' \quad F_z = 2z \quad F_{z'} = 2z'$

$$\begin{cases} -2y'' = 0 \\ 2z - 2z'' = 0 \end{cases} \Rightarrow \begin{array}{l} y' = C_1 \Rightarrow y = C_1x + C_2 \\ \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \Rightarrow z = C_3e^x + C_4e^{-x} \end{array}$$

$$\begin{cases} C_1 + C_2 = 1 \\ 2C_1 + C_2 = 2 \end{cases} \Rightarrow \begin{cases} C_1 = 1 \\ C_2 = 0 \end{cases} \Rightarrow y = x$$

$$\begin{cases} C_3e + C_4e^{-1} = 0 \\ C_3e^2 + C_4e^{-2} = 1 \end{cases} \Rightarrow \begin{cases} C_3e^2 + C_4 = 0 \\ C_3e^4 + C_4 = e^2 \end{cases} \Rightarrow$$

$\Rightarrow C_3(e^4 - e^2) = e^2 \Rightarrow C_3 = \frac{1}{e^2-1}$

$C_4 = -\frac{e^2}{e^2-1}$

$$z = \frac{e^x}{e^2-1} - \frac{e^2 \cdot e^{-x}}{e^2-1} = \frac{e^{x-1} - e^{1-x}}{e^1 - e^{-1}} = \frac{\operatorname{sh}(x-1)}{\operatorname{sh} 1}$$

Ответ: $y = x, \quad z = \frac{\operatorname{sh}(x-1)}{\operatorname{sh} 1}$

№ 8

$$J[y, z] = \int_0^{\pi/2} (y'^2 + z'^2 - 2yz)\,dx \qquad y(0)=0 \quad y(\tfrac{\pi}{2})=1$$
$$z(0)=0 \quad z(\tfrac{\pi}{2})=1$$

$F_y = -2z \quad F_{y'} = 2y' \quad F_z = -2y \quad F_{z'} = 2z'$

$$\begin{cases} -2z - 2y'' = 0 \\ -2y - 2z'' = 0 \end{cases} \Leftrightarrow \begin{cases} y'' + z = 0 \\ z'' + y = 0 \end{cases} \Leftrightarrow \begin{cases} z = -y'' \\ -y^{IV} + y = 0 \end{cases}$$

$-\lambda^4 + 1 = 0 \qquad \lambda^2 = \pm 1 \qquad \lambda = \pm 1; \pm i$

$y = C_1e^x + C_2e^{-x} + C_3\cos x + C_4\sin x$

$y'' = -C_1e^x - C_2e^{-x} + C_3\cos x + C_4\sin x$

$$\begin{cases} C_1 + C_2 + C_3 = 0 \\ C_1e^{\pi/2} + C_2e^{-\pi/2} + C_4 = 1 \\ C_1 + C_2 - C_3 = 0 \\ \ldots \end{cases} \Leftrightarrow \begin{cases} C_3 = 0 \\ C_4 = 1 \\ C_1 = 0 \\ \ldots \end{cases} \qquad y = \sin x \quad z = \sin x$$

$\Phi[y] = \int_{x_0}^{x_1} F(x, y, y')\,dx, \quad y(x_0) = y_0, \; y(x_1) = y_1$

№1

$$J[y]=\int_0^1 yy'^2\,dx \qquad y(0)=1,\ y(1)=\sqrt[3]{4}$$

$$F_y = y'^2 \qquad F_{y'} = 2yy'$$

$$\begin{cases} y'^2 - 2y'^2 - 2yy''=0 \\ y(0)=1 \\ y(1)=\sqrt[3]{4}\end{cases} \iff 2yy''+y'^2=0$$

$$2\frac{dy'}{y'} = -\frac{dy}{y}$$

$$2\int \frac{dy'}{y'} = -\int\frac{dy}{y}$$

$$y' = \frac{C_1}{\sqrt{y}}$$

$$\int\sqrt{y}\,dy = C_1\int dx$$

$$\frac{2}{3}y^{3/2}=C_1x+C_2$$

$$y=(C_1x+C_2)^{2/3}$$

$$\begin{cases} C_2^{2/3}=1 \\ (C_1+C_2)^{2/3}=2^{2/3}\end{cases} \iff \begin{cases} C_2=\pm1 \\ C_1+C_2=2\end{cases} \Rightarrow \left[\begin{array}{l} \begin{cases} C_2=1 \\ C_1=1\end{cases} \\ \begin{cases} C_2=-1 \\ C_1=3\end{cases}\end{array}\right.$$

Ответ: $y_1=(3x-1)^{2/3}$ $\quad y_2=(x+1)^{2/3}$

$$J[y]=\int_0^\pi (4y\cos x + y'^2 - y^2)\,dx \qquad y(0)=0 \qquad y(\pi)=0$$

$$F_y = 4\cos x - 2y \qquad F_{y'}=2y'$$

$$\begin{cases} 4\cos x-2y-2y''=0 \\ y(0)=0 \\ y(\pi)=0\end{cases}$$

$$y''+y=2\cos x$$

I. $\lambda^2+1=0 \Rightarrow \lambda=\pm i \qquad y_{oo}=C_1\cos x + C_2\sin x$

$$y_{ч} = C_1\cos x + C_2 \sin x + x\sin x$$

$$\begin{cases} C_1=0 \\ -C_1=0\end{cases} \Rightarrow y=(C_2+x)\sin x$$

Ответ: $y=(C+x)\sin x$

№3

$$J[y]=\int_0^1 (y'^2-y^2-y)e^{2x}\,dx \qquad y(0)=0 \qquad y(1)=e^{-1}$$

$$F_y=(-2y-1)e^{2x} \qquad F_{y'}=(2y')e^{2x}$$

$$\begin{cases} (-2y-1)e^{2x}-2y''e^{2x}-4y'e^{2x}=0 \\ y(0)=0 \\ y(1)=e^{-1}\end{cases}$$

$$-2y-1-2y''-4y'=0$$

I. $2\lambda^2+4\lambda+2=0$

$\lambda^2+2\lambda+1=0$

$\lambda=-1,\ s=2$

$$y_{oo}=(C_1x+C_2)e^{-x}$$

$$y=(C_1x+C_2)e^{-x}-\frac12 \qquad \begin{cases} C_2-\frac12=0 \\ (C_1+C_2)e^{-1}-\frac12=e^{-1}\end{cases} \iff \begin{cases} C_2=1/2 \\ C_1=\frac12(1+e^{-1})\end{cases}$$

$C_1=\frac12(1+e)$

$$\Phi[y]=\int_{x_0} F(x,y,y')\,dx, \quad y(x_0)=y_0,\ y(x_1)=y_1$$

Ответ: $y = \frac{1}{2}\left(e^{-x} + (1+e)e^{-x} - 1\right)$

№4

$$J[y] = \int_1^e (xy'^2 + yy')\,dx \qquad y(1)=0 \quad y(e)=1$$

$$F_y = y' \qquad F_{y'} = 2xy' + y$$

$$\begin{cases} \cancel{y'} - 2y' - 2xy'' - \cancel{y'} = 0 \\ y(1)=0 \\ y(e)=1 \end{cases}$$

$$xy'' + y' = 0$$

$$x^2y'' + xy' = 0$$

$$\lambda(\lambda-1) + \lambda = 0$$
$$\lambda^2 = 0 \Rightarrow \lambda = 0,\ s=2$$

$$y(t) = C_1 t + C_2$$

$$y(x) = C_1 \ln x + C_2$$

$$\begin{cases} C_2 = 0 \\ C_1 + C_2 = 1 \end{cases} \Rightarrow \begin{cases} C_2 = 0 \\ C_1 = 1 \end{cases}$$

Ответ: $y = \ln x$

№5

$$J[y] = \int_a^b \left(2xy + (x^2+e^y)y'\right)dx \qquad y(a)=A \quad y(b)=B$$

$$F_y = 2x + y'e^y \qquad F_{y'} = x^2 + e^y$$

$$\begin{cases} \cancel{2x} + \cancel{y'e^y} - \cancel{2x} - \cancel{y'e^y} = 0 \\ y(a)=A \\ y(b)=B \end{cases}$$

Ответ: $\varnothing$

$$J[y] = \int_0^1 (y'^2 + y''^2)\,dx \qquad y(0)=0 \quad y'(0)=0$$
$$y(1)=0 \quad y'(1) = -\operatorname{sh}1$$

$$F_y = 0 \qquad F_{y'} = 2y' \qquad F_{y''} = 2y''$$

$$\begin{cases} -2y'' + 2y^{IV} = 0 \\ y(0)=0 \quad y(1)=0 \\ y'(0)=0 \quad y'(1)=-\operatorname{sh}1 \end{cases}$$

$$\lambda^4 - \lambda^2 = 0$$
$$\lambda^2(\lambda^2-1) = 0$$
$$\lambda = 0,\ s=2$$
$$\lambda = \pm 1$$

$$\cancel{y = C_1e^x + C_2e^{-x} + C_3x + C_4,\ y' = C_1e^x - C_2e^{-x} + C_3}$$

$$\begin{cases} C_1 + C_2 + C_4 = 0 \\ C_1e + C_2e^{-1} + C_3 + C_4 = 0 \\ C_1 - C_2 + C_3 = 0 \\ C_1e - C_2e^{-1} + C_3 = -\operatorname{sh}1 \end{cases} \Leftrightarrow \begin{cases} C_4 = -C_1 - C_2 \\ C_3 = C_2 - C_1 \\ C_1e + C_2e^{-1} + C_2 - C_1 - C_1 - C_2 = 0 \\ C_1e - C_2e^{-1} + C_2 - C_1 = -\operatorname{sh}1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} C_4 = -C_1 - C_2 \\ C_3 = C_2 - C_1 \\ -C_1(e-2)e = C_2 \\ C_1(e-1) + C_1(2-e)e(1-e^{-1}) = -\frac{e - e^{-1}}{2} \end{cases}$$

$$C_1\left(e-1 + (2e-e^2)(1-e^{-1})\right) = \frac{e-e^{-1}}{2} \Leftrightarrow$$

$$\Leftrightarrow C_1(e - 1 + 2e - 2 - e^2 + e) = -\frac{e - e^{-1}}{2} \Leftrightarrow$$

$$\Leftrightarrow C_1 =$$

$$y = C_1\operatorname{ch}x + C_2\operatorname{sh}x + C_3x + C_4,\ y' = C_1\operatorname{sh}x + C_2\operatorname{ch}x + C_3$$

$$\begin{cases} C_1 + C_4 = 0 \\ C_1\operatorname{ch}1 + C_2\operatorname{sh}1 + C_3 + C_4 = 0 \\ C_2 + C_3 = 0 \\ C_1\operatorname{sh}1 + C_2\operatorname{ch}1 + C_3 = -\operatorname{sh}1 \end{cases} \Leftrightarrow \begin{cases} C_1 = -C_4 \\ C_2 = -C_3 \\ C_4(1-\operatorname{ch}1) + C_3(1-\operatorname{sh}1) = 0 \\ C_3(1-\operatorname{ch}1) = (C_4-1)\operatorname{sh}1 \end{cases}$$

$$C_3 = \frac{C_4(1-\operatorname{ch}1)}{\operatorname{sh}1 - 1} \Rightarrow \frac{C_4(1-\operatorname{ch}1)^2 - C_4\operatorname{sh}1(\operatorname{sh}1-1)}{\operatorname{sh}1-1} = -\operatorname{sh}1 \Rightarrow$$

$$\Phi[y] = \int_{x_0} F(x,y,y')\,dx, \quad y(x_0)=y_0,\ y(x_1)=y_1$$

$$\Rightarrow C_4(1+\operatorname{ch}^2 1+2\operatorname{ch}1-\operatorname{sh}^2 1+\operatorname{sh}1) = -\operatorname{sh}^2 1+\operatorname{sh}1 \Leftrightarrow$$

$$\Leftrightarrow C_4 = \frac{\operatorname{sh}1-\operatorname{sh}^2 1}{2-2\operatorname{ch}1+\operatorname{sh}1} = \frac{\operatorname{sh}1(1-\operatorname{sh}1)}{2(1-\operatorname{ch}1)+\operatorname{sh}1}$$

$$C_3 = \frac{\operatorname{sh}1(1-\operatorname{ch}1)}{2(1-\operatorname{ch}1)+\operatorname{sh}1}$$

Ответ: $y = -\frac{\operatorname{sh}1(1-\operatorname{sh}1)}{2(1-\operatorname{ch}1)+\operatorname{sh}1}\operatorname{ch}x - \frac{\operatorname{sh}1(1-\operatorname{ch}1)}{2(1-\operatorname{ch}1)+\operatorname{sh}1}\operatorname{sh}x + \frac{\operatorname{sh}1(1-\operatorname{ch}1)}{2(1-\operatorname{ch}1)+\operatorname{sh}1}x + \frac{\operatorname{sh}1(1-\operatorname{sh}1)}{2(1-\operatorname{ch}1)+\operatorname{sh}1}$

№7

$$J[y] = \frac{1}{2}\int_0^1 (y'')^2 dx \quad y(0)=0,\ y'(0)=0,\ y(1)=0{,}5,\ y'(1)=1$$

$F_y = 0 \quad F_{y'} = 0 \quad F_{y''} = y''$

$-y^{IV} = 0 \Leftrightarrow y''' = 6C_1 \Rightarrow y'' = 6C_1x+2C_2 \Rightarrow y' = 3C_1x^2+2C_2x+C_3$

$\Rightarrow y = C_1x^3+C_2x^2+C_3x+C_4,\ y' = 3C_1x^2+2C_2x+C_3$

$$\begin{cases} C_4 = 0 \\ C_3 = 0 \\ C_1+C_2+C_3+C_4 = 0{,}5 \\ 3C_1+2C_2+C_3 = 1 \end{cases} \Leftrightarrow \begin{cases} C_4 = C_3 = 0 \\ C_2 = 0{,}5 - C_1 \\ C_1 = 0 \end{cases} \Rightarrow \begin{cases} C_1 = C_3 = C_4 = 0 \\ C_2 = 0{,}5 \end{cases}$$

Ответ: $y = 0{,}5x^2$

№8

$$J[y] = \int_{-1}^{0} (240y - y'''^2)\,dx \quad \begin{matrix} y(-1)=1 & y'(-1)=-4{,}5 & y''(-1)=16 \\ y(0)=0 & y'(0)=0 & y''(0)=0 \end{matrix}$$

$F_y = 240 \quad F_{y'} = 0 \quad F_{y''} = 0 \quad F_{y'''} = -2y'''$

$240 + 2y^{VI} = 0 \Leftrightarrow y^{VI} = -120 \Rightarrow$

$\Rightarrow y^{V} = -120x + 120C_1 \Rightarrow y^{IV} = -60x^2+120C_1x+8C_2 \Rightarrow$

$\Rightarrow y''' = -20x^3+60C_1x^2+8C_2x+C_3 \Rightarrow$

$\Rightarrow y'' = -5x^4+20C_1x^3+8C_2x^2+6C_3x+2C_4 \Rightarrow$

$\Rightarrow y' = -x^5+5C_1x^4+4C_2x^3+3C_3x^2+2C_4x+C_5 \Rightarrow$

$\Rightarrow y = -\frac{x^6}{6}+C_1x^5+C_2x^4+C_3x^3+C_4x^2+C_5x+C_6$

$$\begin{cases} C_6 = 0 \\ C_5 = 0 \\ C_4 = 0 \\ -\frac{1}{6} - C_1 + C_2 - C_3 = 1 \\ 1 - 4C_2 + 3C_3 = -4{,}5 - 5C_1 \\ -5 - 20C_1 + 8C_2 - 6C_3 = 16 \end{cases} \Leftrightarrow \begin{cases} C_1 = C_2 - C_3 - \frac{7}{6} \\ 3C_3 - 4C_2 = -5{,}5 - 5C_2 + 5C_3 + \frac{35}{6} \\ 8C_2 - 6C_3 = 21 + 20C_2 - 20C_3 - \frac{140}{6} \end{cases}$$

$$\Rightarrow \begin{cases} C_1 = C_2 - C_3 - \frac{7}{6} \\ C_2 = 2C_3 + 1/3 \\ 14C_3 = 24C_3 + 4 + 21 - \frac{70}{3} \end{cases} \Rightarrow \begin{cases} C_3 = -1/6 \\ C_2 = 0 \\ C_1 = -1 \end{cases}$$

Ответ: $y = -\frac{x^3}{6}(x^3+6x^2+1)$

№9

$$J[y,z] = \int_0^{\pi} (2yz - 2y^2 + y'^2 - z'^2)\,dx \quad y(0)=0,\ y(\pi)=1,\ z(0)=0,\ z(\pi)=-1$$

$F_y = 2z - 4y \quad F_{y'} = 2y' \quad F_z = 2y \quad F_{z'} = -2z'$

$$\begin{cases} 2z - 4y - 2y'' = 0 \\ 2y + 2z'' = 0 \end{cases} \Leftrightarrow \begin{cases} z = 2y + y'' \\ y + (2y+y'')'' = 0 \end{cases}$$

$y + 2y'' + y^{IV} = 0 \qquad \lambda^4 + 2\lambda^2 + 1 = 0$

$(\lambda^2+1)^2 = 0$

$\lambda = \pm i, \quad s = 2$

$y = (C_1x+C_2)\cos x + (C_3x+C_4)\sin x$

$y' = C_1\cos x - (C_1x+C_2)\sin x + C_3\sin x + (C_3x+C_4)\cos x$

$y'' = -C_1\sin x - C_1\sin x - (C_1x+C_2)\cos x + C_3\cos x + C_3\cos x -$

$- (C_3x+C_4)\sin x =$

$= (-2C_1 - C_3x - C_4)\sin x + (-C_1x - C_2 + 2C_3)\cos x$

$z = 2y + y'' = (C_1x + C_2 + 2C_3)\cos x + (C_3x + C_4 - 2C_1)\sin x$

$$\begin{cases} C_2 = 0 \\ -C_1\pi = 1 \\ C_2 + 2C_3 = 0 \\ -C_1\pi + C_2 + 2C_3 = -1 \end{cases} \Leftrightarrow \begin{cases} C_2 = 0 \\ C_3 = 0 \\ C_1 = -\frac{1}{\pi} \\ C_3 = 1 \end{cases}$$

Ответ: $y = C\sin x - \frac{x}{\pi}\cos x$

$z = C\sin x + \frac{1}{\pi}(2\sin x - x\cos x)$

№10

$J[y,z] = \int_0^1 (y'^2 + z'^2 + 2y)\,dx \quad y(0) = 1,\ y(1) = \frac{3}{2}$

$z(0) = 0,\ z(1) = 1$

$F_y = 2 \quad F_{y'} = 2y' \quad F_z = 0 \quad F_{z'} = 2z'$

$$\begin{cases} 2 - 2y'' = 0 \\ -2z'' = 0 \end{cases} \Leftrightarrow \begin{cases} y'' = 1 \\ z'' = 0 \end{cases} \Leftrightarrow \begin{cases} y = \frac{x^2}{2} + C_3x + C_4 \\ z = C_1x + C_2 \end{cases}$$

$$\begin{cases} C_4 = 1 \\ \frac{1}{2} + C_3 + C_4 = 3/2 \\ C_2 = 0 \\ C_1 + C_2 = 1 \end{cases} \Leftrightarrow \begin{cases} C_1 = 1 \\ C_2 = 0 \\ C_3 = 0 \\ C_4 = 1 \end{cases}$$

Ответ: $y = \frac{x^2}{2} + 1$; $z = x$

$y = f(x) \qquad y = f_1(x)$

$\rho = \rho[f(x), f_1(x)] = \max_{x\in[a,b]} |f(x) - f_1(x)|$

$\rho_n = \max_{0\le k\le n} \max_{x\in[a,b]} |f^{(k)}(x) - f_1^{(k)}(x)|$

ε-окрестность нулевого порядка $\rho_0[f(x), f_1(x)] < \varepsilon$ - сильная

первого порядка $\rho_1[f(x), f_1(x)] < \varepsilon$ - слабая

$\Phi[y(x)]$ достигает на $y = y_0(x)$ максимума, если

$\Phi[y_0(x)] \ge \Phi[y(x)] \quad \forall y(x)$ из ε-окрестн.

$\Phi[y] = \int_{x_0}^{x_1} F(x,y,y')\,dx \quad y(x_0) = y_0,\ y(x_1) = y_1$

Дост. усл. Лежандра

$F_{y'y'} > 0$

№1

$\Phi[y] = \int_0^1 e^x(y^2 + \frac{1}{2}y'^2)\,dx - \quad y(0) = 1 \quad y(1) = e$

$F_y = 2e^xy \quad F_{y'} = e^xy'$

$2e^xy - e^xy' - e^xy'' = 0 \Rightarrow y'' + y' - 2y = 0$

$\Phi[y] = \int_{x_0}^{x_1} F(x, y, y')\,dx, \quad y(x_0) = y_0,\ y(x_1) = y_1$

$\lambda = -2;\ 1$

$y = C_1 e^{-2x} + C_2 e^{x}$

$$\begin{cases} C_1 + C_2 = 1 \\ C_1 e^{-2} + C_2 e = e \end{cases} \Leftrightarrow \begin{cases} C_1 = 1 - C_2 \\ 1 - C_2 + C_2 e^3 = e^3 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} C_2 = 1 \\ C_1 = 0 \end{cases} \qquad y = e^{x}$$

$F_{y'y'} = e^x > 0$ (от y независ.)

Ответ: e^x, сильн. мин.

№3

$\Phi[y] = \int_1^2 \frac{x^3}{y'^2}\,dx \qquad y(1) = 1,\ y(2) = 4$

$F_y = 0 \qquad F_{y'} = -\frac{2x^3}{y'^3}$

$$\frac{6x^2}{y'^3} + \frac{6x^3}{y'^4} = 0 \Rightarrow \frac{x^3}{y'^3} = \frac{1}{8C_1^3}$$

$$y'^3 = 8C_1^3 x^3 \Rightarrow y' = 2C_1 x \Rightarrow y = C_1 x^2 + C_2$$

$$\begin{cases} C_1 + C_2 = 1 \\ 4C_1 + C_2 = 4 \end{cases} \Rightarrow \begin{cases} C_1 = 1 \\ C_2 = 0 \end{cases} \Rightarrow y = x^2$$

$F_{y'y'} = \frac{6x^3}{y'^4} > 0$ – сил. мин

Ответ: $y = x^2$ – сил. мин.

№5

$\Phi[y] = \int_0^1 (1+x)\,y'^2\,dx \qquad y(0) = 0 \quad y(1) = 1$

$F_y = 0 \qquad F_{y'} = 2(1+x)y'$

$-2y' - 2(1+x)y'' = 0 \quad \Leftrightarrow \quad (1+x)y'' + y' = 0$

$(1+x)y' = C_1 \Leftrightarrow y' = \frac{C_1}{1+x} \Rightarrow y = C_1 \ln(1+x) + C_2$

$$\begin{cases} C_2 = 0 \\ C_1 \ln 2 + C_2 = 1 \end{cases} \Leftrightarrow \begin{cases} C_2 = 0 \\ C_1 = 1/\ln 2 \end{cases} \Rightarrow y = \log_2(1+x)$$

$F_{y'y'} = 2(1+x) > 0$ – сильн. мин.

Ответ: $y = \log_2(1+x)$ – сильн. мин.

№10

$\Phi[y] = \int_1^2 (x^2 y'^2 + 12y^2)\,dx \qquad y(1) = 1 \quad y(2) = 8$

$F_y = 24y \qquad F_{y'} = 2x^2 y'$

$24y - 4xy' - 2x^2 y'' = 0$

$x^2 y'' + 2xy' - 12y = 0$

$\lambda(\lambda - 1) + 2\lambda - 12 = 0$

$\lambda^2 + \lambda - 12 = 0$

$\lambda = -4;\ 3$

$y(t) = C_1 e^{-4t} + C_2 e^{3t} \quad ; \quad y(x) = C_1 x^{-4} + C_2 x^3$

$\Phi[y] = \int_{x_0}^{x_1} F(x, y, y')\,dx, \quad y(x_0) = y_0,\ y(x_1) = y_1$

$$\begin{cases} C_1 + C_2 = 1 \\ \dfrac{C_1}{16} + C_2 \cdot 8 = 8 \end{cases} \Leftrightarrow \begin{cases} C_1 = 0 \\ C_2 = 1 \end{cases} \qquad y(x) = x^3$$

$F_{y'y'} = 2x^2 > 0$ – сильн. мин.

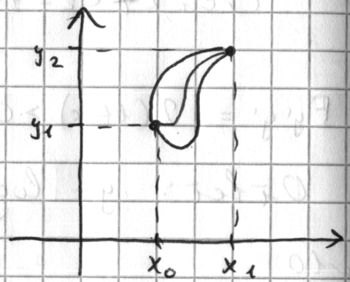
Ответ: $y(x) = x^3$ – сильн. мин.

Изопериметрические задачи

$$(*)\ \Phi[y] = \int_{x_0}^{x_1} F(x, y, y')\,dx \qquad y(x_0) = y_0 \qquad y(x_1) = y_1$$

$$? \quad \varphi(x, y, y') = C_i$$

$$\int_{x_0}^{x_1} \sqrt{1 + f'^2(x)}\,dx = l$$



$$\left[\begin{array}{l} (*) \\ \displaystyle\int_{x_0}^{x_1} f_i(x, y, y')\,dx = l_i, \quad i = \overline{1, n} \end{array}\right.$$

$$F^*(x, y, y') = F(x, y, y') + \sum_{i=1}^{n} \lambda_i f_i(x, y, y')$$

$\Rightarrow$ ур-ние Эйлера для F^*

$$\Phi[y] = \int_0^1 y'^2\,dx \qquad y(0) = 1 \qquad y(1) = 6 \qquad \int_0^1 y\,dx = 3$$

$$F^*(x, y, y') = y'^2 + \lambda y$$

$$F^*_y = \lambda \qquad F^*_{y'} = 2y'$$

$$\begin{cases} \lambda - 2y'' = 0 \\ y(0) = 1 \\ y(1) = 6 \end{cases} \Rightarrow \quad y'' = \frac{\lambda}{2} \Rightarrow y' = \frac{\lambda}{2}x + C_1$$

$$y = \frac{\lambda x^2}{4} + C_1 x + C_2$$

$$\begin{cases} C_2 = 1 \\ \dfrac{\lambda}{4} + C_1 + C_2 = 6 \\ \displaystyle\int_0^1 \left(\frac{\lambda x^2}{4} + C_1 x + C_2\right) dx = 3 \end{cases} \Leftrightarrow \begin{cases} C_2 = 1 \\ \lambda + 4C_1 = 20 \\ \left.\left(\dfrac{\lambda x^3}{12} + \dfrac{C_1 x^2}{2} + C_2 x\right)\right|_0^1 = 3 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} C_2 = 1 \\ \lambda = 20 - 4C_1 \\ \dfrac{\lambda}{12} + \dfrac{C_1}{2} + 1 = 3 \end{cases} \Leftrightarrow \begin{cases} C_2 = 1 \\ \lambda = 20 - 4C_1 \\ \lambda = 24 - 6C_1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} C_2 = 1 \\ C_1 = 2 \\ \lambda = 12 \end{cases}$$

$$y = 3x^2 + 2x + 1$$

Ответ: $y = 3x^2 + 2x + 1$

$\Phi[y] = \int_{x_0}^{x_1} F(x, y, y')\,dx, \quad y(x_0) = y_0,\ y(x_1) = y_1$

$$\Phi[y] = \int_0^1 (x^2 + y'^2)\,dx \qquad y(0) = 0 \quad y(1) = 0$$

$$\int_0^1 y^2\,dx = 2$$

$$F^* = x^2 + y'^2 + \lambda y^2$$

$$F^*_y = 2\lambda y \qquad F^*_{y'} = 2y'$$

$$2\lambda y - 2y'' = 0$$

$$y'' - \lambda y = 0$$

$$\alpha^2 - \lambda = 0$$

$$\cancel{\alpha = \pm\sqrt{\lambda}}$$

1) $\lambda = 0 \qquad \begin{cases} y'' = C_1 x + C_2 \\ y(0) = 0 \\ y(1) = 0 \end{cases} \Rightarrow y \equiv 0$

2) $\lambda > 0, \ \lambda = \mu^2, \ \mu > 0$

$$y = C_1 e^{\mu x} + C_2 e^{-\mu x}$$

$$\begin{cases} C_1 + C_2 = 0 \\ C_1 e^{\mu} + C_2 e^{-\mu} = 0 \end{cases} \quad \begin{cases} C_2 = -C_1 \\ C_1 e^{2\mu} - C_1 = 0 \end{cases} \Leftrightarrow C_1 = C_2 = 0 \Rightarrow y \equiv 0$$

3) $\lambda < 0 \Rightarrow \lambda = -\mu^2, \mu > 0$

$$y = C_1 \cos\mu x + C_2 \sin\mu x$$

$$\begin{cases} C_1 = 0 \\ C_1\cos\mu + C_2\sin\mu = 0 \end{cases} \Leftrightarrow \mu_n = \pi n, \ n \in \mathbb{N}$$

$$y_n = C_2 \sin \pi n x, \ n \in \mathbb{N}$$

$$\int_0^1 C_2^2 \sin^2 \pi n x\,dx = C_2^2 \int_0^1 \frac{1 - \cos 2\pi n x}{2}\,dx =$$

$$= C_2^2\left(\frac{1}{2} - \frac{1}{2}\int_0^1 \cos 2\pi n x\,dx\right) = \frac{C_2^2}{2} - \frac{C_2^2}{4\pi n}\int_0^1 \cos 2\pi n x\,d2\pi n x =$$

$$= \frac{C_2^2}{2} - \frac{C_2^2}{4\pi n}\sin 2\pi n x\Big|_0^1 = \frac{C_2^2}{2} = 2 \Rightarrow C_2^2 = 4 \Rightarrow$$

$$\Rightarrow C_2 = \pm 2$$

Ответ: $y(x) = \pm 2 \sin \pi n x, \ n \in \mathbb{N}$

Прод. по пар., дифф. в инт (по пар., по x), ~~(нач. сост.~~, ~~зад. Коши~~, задача Коши для ур. м. в пост. произв. + вариац. исч., изопер.

Д.р

№ 2

$$\Phi[y] = \int_0^1 e^y y'^2\,dx \qquad y(0) = 0, \ y(1) = \ln 4$$

$$\cancel{F_y = 0} \qquad \cancel{F_{y'} = 2e^x y'}$$

$$\cancel{\frac{d}{dx}(2e^x y') = 0 \Rightarrow e^x y' = C_1 \Rightarrow y = -C_1 e^{-x} + C_2}$$

$$\cancel{\begin{cases} -C_1 + C_2 = 0 \\ -C_1 \cdot \frac{1}{e} + C_2 = \ln 4 \end{cases}}$$

$$F_y = e^y y'^2 \qquad F_{y'} = 2e^y y'$$

$$e^y y'^2 - 2e^y y'' - 2e^y y'^2 = 0$$

$$y'^2 + 2y'' = 0 \Rightarrow$$

$$\Phi[y] = \int_{x_0}^{x_1} F(x, y, y')\,dx, \quad y(x_0) = y_0, \ y(x_1) = y_1$$

$$-\frac{2}{y'} = -x + c_1$$

$$y' = \frac{2}{x+c_1}$$

$$\int dy = \int \frac{2\,dx}{x+c_1} \Rightarrow y = 2\ln(x+c_1) + c_2$$

$$\begin{cases} 2\ln c_1 + c_2 = 0 \\ 2\ln(1+c_1) + c_2 = \ln 4 \end{cases} \Leftrightarrow \begin{cases} c_2 = -2\ln c_1 \\ \ln\frac{1+c_1}{c_1} = \ln 2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} c_1 = 1 \\ c_2 = 0 \end{cases} \Rightarrow y = 2\ln(x+1)$$

$F_{y'y'} = 2e^y > 0 \Rightarrow$ сильн. мин.

Ответ: $y = 2\ln(1+x)$, сильн. мин.

№4

$$\Phi[y] = \int_0^a \frac{dx}{y'} \qquad y(0)=0,\ y(a)=b,\ b>0$$

$$F_y = 0 \qquad F_{y'} = -\frac{1}{y'^2}$$

$$\frac{d}{dx}\left(-\frac{1}{y'^2}\right) = 0 \Rightarrow \frac{1}{y'^2} = \frac{1}{c_1^2} \Leftrightarrow y' = c_1 \Rightarrow y = c_1x + c_2$$

$$\begin{cases} c_2 = 0 \\ c_1 a = b \end{cases} \Rightarrow \begin{cases} c_1 = \frac{b}{a} \\ c_2 = 0 \end{cases} \Rightarrow y = \frac{bx}{a}$$

$F_{y'y'} = \frac{2}{y'^3} > 0 \Rightarrow$ сил. мин.

Ответ: $y = \frac{bx}{a}$ сил. мин.

№6

$$\Phi[y] = \int_0^{\pi/2} (y^2 - y'^2)\,dx \qquad y(0) = 1,\ y(\pi/2) = 1$$

$$F_y = 2y \qquad F_{y'} = -2y'$$

$$2y + 2y'' = 0 \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

$$y = c_1\cos x + c_2 \sin x$$

$$\begin{cases} c_1 = 1 \\ c_2 = 1 \end{cases} \Rightarrow y = \cos x + \sin x$$

$F_{y'y'} = -2 < 0 \Rightarrow$ сильн. макс.

Ответ: $y = \cos x + \sin x$, сильн. макс.

№7

$$\Phi[y] = \int_{-1}^{1} (y'^3 + y'^2)\,dx \qquad y(-1) = -1,\ y(1) = 3$$

$$F_y = 0 \qquad F_{y'} = 3y'^2 + 2y'$$

$$3y'^2 + 2y' = c_1$$

$$y' = -\frac{1}{3} \pm \frac{1}{3}\sqrt{3c_1+1} \Rightarrow y = \left(-\frac{1}{3} \pm \frac{1}{3}\sqrt{3c_1+1}\right)x + c_2$$

$$\begin{cases} 3c_2 + 1 + \sqrt{3c_1+1} = -3 \\ 3c_2 - 1 - \sqrt{3c_1+1} = 9 \end{cases} \Leftrightarrow \varnothing$$

$$\begin{cases} 3c_2 + 1 - \sqrt{3c_1+1} = -3 \\ 3c_2 - 1 + \sqrt{3c_1+1} = 9 \end{cases} \Leftrightarrow \begin{cases} c_2 = 1 \\ \sqrt{3c_1+1} = 7 \end{cases}$$

$$y = \frac{1}{3}(-1+7)x + 1 = 2x + 1$$

$$\Phi[y] = \int_{x_0}^{x_1} F(x, y, y')\,dx, \quad y(x_0) = y_0,\ y(x_1) = y_1$$

$F_{y'y'} = 2 > 0$ сильн. мин.

Ответ: $y = x^3 - 1$, сильн. мин.

№15

$\Phi[y] = \int_0^1 y'^2\,dx \quad y(0) = 0 \quad y(1) = 0.25 \quad \int_0^1 (y - y'^2)\,dx = \frac{1}{12}$

$F^*(x, y, y') = y'^2 + \lambda(y - y'^2)$

$F^*_y = \lambda \qquad F^*_{y'} = 2y' - 2\lambda y' = (2 - 2\lambda)y'$

$\lambda - (2 - 2\lambda)y'' = 0 \Leftrightarrow (2 - 2\lambda)y'' - \lambda = 0$

$\Rightarrow y'' = \frac{\lambda}{2 - 2\lambda} \Leftrightarrow$

$\Leftrightarrow y' = \frac{\lambda}{2 - 2\lambda}x + C_1 \Leftrightarrow y = \frac{\lambda}{4 - 4\lambda}x^2 + C_1 x + C_2$

$$\begin{cases} C_2 = 0 \\ \frac{\lambda}{4 - 4\lambda} + C_1 + C_2 = \frac{1}{4} \end{cases} \Leftrightarrow \begin{cases} C_2 = 0 \\ C_1 = \frac{1}{4} + \frac{\lambda}{4\lambda - 4} \end{cases}$$

$$\int_0^1 \left(-\frac{\lambda}{4\lambda - 4}x^2 + \frac{1}{4}x + \frac{\lambda}{4\lambda - 4}x - \left(-\frac{\lambda}{2\lambda - 2}x + \frac{1}{4} + \frac{\lambda}{4\lambda - 4} \right)^2 \right) dx =$$

$$= -\frac{\lambda}{4\lambda - 4} \cdot \frac{1}{3} + \left(\frac{\lambda}{4\lambda - 4} + \frac{1}{4} \right) \cdot \frac{1}{2} -$$

$$- \int_0^1 \left[\left(\frac{\lambda}{4\lambda - 4} + \frac{1}{4} \right)^2 + \frac{\lambda^2}{4(\lambda - 1)^2}x^2 - \frac{\lambda}{\lambda - 1}\left(\frac{\lambda}{4(\lambda - 1)} + \frac{1}{4} \right)x \right] dx =$$

$$= \frac{\ldots}{48(\lambda - 1)} + \frac{\ldots}{48(\lambda - 1)} - \frac{(2\lambda - 1)^2}{16(\lambda - 1)^2} - \frac{\ldots}{12(\lambda - 1)^2} + \frac{\lambda(2\lambda - 1)}{8(\lambda - 1)^2} =$$

$$= \frac{(8\lambda - 6)(\lambda - 1)}{48(\lambda - 1)^2} - \frac{12\lambda^2 + 3 - 12\lambda}{48(\lambda - 1)^2} - \frac{4\lambda^2}{48(\lambda - 1)^2} +$$

$$+ \frac{12\lambda^2 - 6\lambda}{48(\lambda - 1)^2} = \frac{1}{12} \Leftrightarrow$$

$\Leftrightarrow 8\lambda^2 - 8\lambda - 6\lambda + 6 - 16\lambda^2 + 12\lambda - 3 + 12\lambda^2 - 6\lambda = 4(\lambda - 1)^2 \Leftrightarrow$

$\Leftrightarrow \cancel{4\lambda^2} - \cancel{8\lambda} + 3 = \cancel{4\lambda^2} - \cancel{8\lambda} + 4$

Пусть $y = f(x)$ и $y = f_1(x)$ непрерывны на $[a, b]$.

Расстояние нулевого порядка между кривыми $y = f(x)$ и $y = f_1(x)$:

$$\rho = \rho[f(x), f_1(x)] = \max_{a \le x \le b} |f(x) - f_1(x)|.$$

Пусть $y = f(x)$ и $y = f_1(x)$ имеют на отрезке $[a, b]$ непрерывные производные до порядка n включительно.

Расстояние *n*-го порядка между кривыми $y = f(x)$ и $y = f_1(x)$:

$$\rho_n = \rho_n[f(x), f_1(x)] = \max_{0 \le k \le n} \max_{a \le x \le b} \left|f^{(k)}(x) - f_1^{(k)}(x)\right|.$$

ε – окрестностью *n*-го порядка кривой $y = f(x)$ $(a \le x \le b)$ называется совокупность кривых $y = f_1(x)$, расстояния *n*-го порядка которых до кривой $y = f(x)$ меньше ε :

$$\rho_n = \rho_n[f(x), f_1(x)] < \varepsilon.$$

ε – окрестность нулевого порядка называют **сильной ε – окрестностью функции $y = f(x)$.**

ε – окрестность первого порядка называют **слабой ε – окрестностью функции $y = f(x)$.**

Функционал $\Phi[y(x)]$ достигает на кривой $y = y_0(x)$ **сильного относительного максимума (минимума)**, если для всех допустимых кривых $y = y(x)$, расположенных в некоторой ε – окрестности нулевого порядка кривой $y = y_0(x)$, имеем

$$\Phi[y(x)] \le \Phi[y_0(x)] \quad (\Phi[y(x)] \ge \Phi[y_0(x)]).$$

Функционал $\Phi[y(x)]$ достигает на кривой $y = y_0(x)$ **слабого относительного максимума (минимума)**, если для всех допустимых кривых $y = y(x)$, расположенных в некоторой ε – окрестности первого порядка кривой $y = y_0(x)$, имеем

$$\Phi[y(x)] \le \Phi[y_0(x)] \quad (\Phi[y(x)] \ge \Phi[y_0(x)]).$$

Всякий сильный экстремум есть в то же время и слабый, но не наоборот.

Экстремум функционала $\Phi[y(x)]$ на всей совокупности функций, на которых он определён, называется абсолютным экстремумом.

Для простейшей вариационной задачи для функционала

$$\Phi[y] = \int_{x_0}^{x_1} F(x, y, y')dx, \quad y(x_0) = y_0,\ y(x_1) = y_1$$

Достаточные условия Лежандра экстремума функционала. Пусть функция $F(x, y, y')$ имеет непрерывную частную производную $F_{y'y'}(x, y, y')$ и пусть экстремаль C включена в поле экстремалей.

Если на экстремали C имеем $F_{y'y'} > 0$ (< 0), то на кривой C достигается слабый минимум (максимум) функционала.

В том случае, когда $F_{y'y'} \ge 0$ (≤ 0) в точках *(x,y)*, близких к экстремали C, при произвольных значениях y', то имеем сильный минимум (максимум).

Исследовать на экстремум функционалы:

1. $\Phi[y]=\int\limits_0^1 e^x\left(y^2+\frac{1}{2}y'^2\right)dx,\quad y(0)=1,\ y(1)=e$ $\qquad y=e^x$, сильный мин.

2. $\Phi[y]=\int\limits_0^1 e^x y'^2\,dx,\quad y(0)=1,\ y(1)=\ln 4$ $\qquad y=2\ln(x+1)$, сильный мин.

3. $\Phi[y]=\int\limits_1^2 \frac{x^3}{y'^2}\,dx,\quad y(1)=1,\ y(2)=4$ $\qquad y=x^2$, сл. мин.

4. $\Phi[y]=\int\limits_0^a \frac{1}{y'}\,dx,\quad y(0)=0,\ y(a)=b,\ b>0$ $\qquad y=\frac{bx}{a}$, сл. мин.

5. $\Phi[y]=\int\limits_0^1 (1+x)y'^2\,dx,\quad y(0)=0,\ y(1)=1$ $\qquad y=\log_2(1+x)$, сильный мин.

6. $\Phi[y]=\int\limits_0^{\pi/2}\left(y^2-y'^2\right)dx,\quad y(0)=1,\ y(\pi/2)=1$ $\qquad y=\cos x+\sin x$, сильный макс.

7. $\Phi[y]=\int\limits_{-1}^1\left(y'^3+y^2\right)dx,\quad y(-1)=-1,\ y(1)=3$ $\qquad y=2x+1$, сл. мин.

8. $\Phi[y]=\int\limits_0^2\left(xy'+y'^2\right)dx,\quad y(0)=1,\ y(2)=0$ $\qquad y=-\frac{x^2}{4}+1$, сильный мин.

9. $\Phi[y]=\int\limits_0^{\pi/4}\left(4y^2-y'^2+8y\right)dx,\quad y(0)=-1,\ y(\pi/4)=0$ $\qquad y=\sin 2x-1$, сильный макс.

10. $\Phi[y]=\int\limits_1^2\left(x^2y'^2+12y^2\right)dx,\quad y(1)=1,\ y(2)=8$ $\qquad y=x^3$, сильный мин.

11. $\Phi[y]=\int\limits_0^{\pi/4}\left(y^2-y'^2+6y\sin 2x\right)dx,\quad y(0)=0,\ y(\pi/4)=1$ $\qquad y=\sin 2x$, сильный макс.

12. $\Phi[y]=\int\limits_1^3\left(12xy+y'^2\right)dx,\quad y(1)=0,\ y(3)=26$ $\qquad y=x^3-1$, сильный макс. ?

Изопериметрические задачи

Найти экстремали

13. $\Phi[y]=\int_0^1 y'^2dx,\quad y(0)=1,\quad y(1)=6$, при условии $\int_0^1 ydx=3$. Ответ: $y=3x^2+2x+1$

14. $\Phi[y]=\int_0^1(x^2+y'^2)dx,\quad y(0)=0,\quad y(1)=0$, при условии $\int_0^1 y^2dx=2$. Ответ: $y=\pm 2\sin\pi n$

15. $\Phi[y]=\int_0^1 y'^2dx,\quad y(0)=1,\quad y(1)=0.25$, при условии $\int_0^1(y-y'^2)dx=\frac{1}{12}$. Ответ: $y=x^2/4$.