□ До д-во на Теор. 16

$\forall \varepsilon > 0\ \exists N,\ \exists \delta > 0,\ \delta < \pi$:

$|S_n(x_0, f) - \tilde{f}(x_0)| < \left| \int_{-\delta}^{\delta} (f(x_0+t) - \tilde{f}(x_0)) \cdot D_n(t)\,dt \right| + \frac{2}{3}\varepsilon$. (1)

$\exists \delta_0 > 0$ усл. Липш. справа (M_1, α_1), слева (M_2, α_2)

$M = \max(M_1, M_2)$, $\alpha = \min(\alpha_1, \alpha_2)$

$\Rightarrow |f(x_0+t) - f(x_0+0)| \le M t^\alpha$, $\forall t \in (0, \delta_0)$

$|f(x_0+t) - f(x_0-0)| \le M |t|^\alpha$, $\forall t \in (-\delta_0, 0)$ (2)

Нека δ: $0 < \delta < \min(\pi, \delta_0)$ в ф-ла (1)

$\int_{-\delta}^{\delta} D_n(t)\,dt = 2\int_0^{\delta} D_n(t)\,dt = 2\int_{-\delta}^{0} D_n(t)\,dt$.

$\tilde{f}(x_0) \int_{-\delta}^{\delta} D_n(t)\,dt = \frac{f(x_0+0) + f(x_0-0)}{2} \int_{-\delta}^{\delta} D_n(t)\,dt =$

$= \frac{f(x_0+0)}{2} \cdot 2 \int_0^{\delta} D_n(t)\,dt + \frac{f(x_0-0)}{2} \cdot 2 \cdot \int_{-\delta}^{0} D_n(t)\,dt$

$\left| \int_0^{\delta} (f(x_0+t) - \tilde{f}(x_0)) D_n(t)\,dt \right| = \left| \int_0^{\delta} (f(x_0+t) - f(x_0+0)) D_n(t)\,dt + \int_{-\delta}^{0} (f(x_0+t) - f(x_0-0)) D_n(t)\,dt \right| \le$

$\le \int_0^{\delta} |f(x_0+t) - f(x_0+0)| \frac{1}{\pi} \left| \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}} \right| dt + \int_{-\delta}^{0} |f(x_0+t) - f(x_0-0)| \frac{1}{\pi} \frac{|\sin(n+\frac{1}{2})t|}{2|\sin\frac{t}{2}|}\,dt$ (≤)

$|\sin x| \ge \frac{2}{\pi}|x|$, $\forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ (изследване $\frac{\sin x}{x}$)

$|\sin\frac{t}{2}| \ge \frac{2}{\pi} \frac{|t|}{2} = \frac{|t|}{\pi}$

$\forall t \in (-\pi, \pi)$ $\frac{t}{2} \in (-\frac{\pi}{2}, \frac{\pi}{2})$

(≤) $\frac{M}{2\pi} \cdot \pi \left(\int_0^{\delta} t^\alpha \cdot \frac{1}{t}\,dt + \int_{-\delta}^{0} |t|^\alpha \cdot \frac{1}{|t|}\,dt \right) =$

$= \frac{M}{2} \cdot 2 \int_0^{\delta} t^{\alpha-1}\,dt = M \int_0^{\delta} t^{\alpha-1}\,dt = \frac{M\delta^\alpha}{\alpha} < \frac{\varepsilon}{3}$

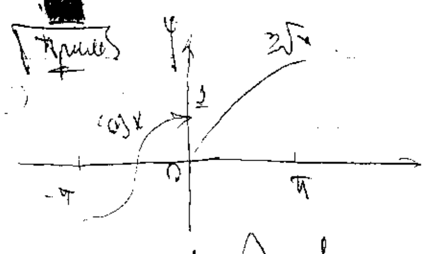
усл. на $\delta = \delta(\varepsilon)$

$0 < \delta < \left(\frac{\alpha\varepsilon}{3M}\right)^{\frac{1}{\alpha}} < \pi, \delta_0$

$\overset{\alpha > 0}{\underset{\text{нач. в (1)}}{\Rightarrow}}$ $|S_n(x_0, f) - \tilde{f}(x_0)| < \frac{\varepsilon}{3} + \frac{2}{3}\varepsilon = \varepsilon$

$\forall n > N(\varepsilon)$, т.е. $S_n(x_0, f) \xrightarrow[n\to\infty]{} \tilde{f}(x_0)$

Пример



$S_n(0) \to \frac{1}{2}$.

$x \ne 0$ $\sqrt[3]{x}$ $\exists (\sqrt[3]{x})'$

$x \in (0, \pi)$, $\alpha = 1$
$x \in (-\pi, 0)$

$\exists (\cos x)'$, $\alpha = 1$

$x = 0$ $\cos x$, $\alpha = 1$

$\sqrt[3]{x}$ $\alpha = \frac{1}{3}$.